“Nike’s Trick”

I don’t remember why this is called Nike’s trick, but it is one way to fill in the gap in the proof in Hartshorne that every open affine subscheme of a locally noetherian scheme is the spectrum of a noetherian ring. It is also one way to prove all similar statements (see exercises II.3.1-5 or so). Also, I’m overly precise in the proof, but I think it fits the tone of Hartshorne II.3

Theorem. *(Nike’s Trick) Let $U_i = Spec(A_i)$ for $i = 1, 2$ be two open affine subschemes of a scheme X. Then, for any $x \in U_1 \cap U_2$, there exists an open subscheme V such that $x \in V \subset U_1 \cap U_2$ and V is a basic open affine for both U_1 and U_2. (To be pedantic about it, there exist elements $f_i \in A_i$ $i = 1, 2$ such that the induced morphisms $Spec(A_{i,f_i}) \to Spec(A_i)$ are isomorphisms onto V.)*

Proof. The proof is essentially ring theoretic. First of all, since the basic open affines form a base for the topology of U_1, we may find $f \in A_1$ such that $D(f) \subset U_1 \cap U_2$. Since a basic open affine of a basic open affine is a basic open affine, we may thus replace U_1 with $D(f)$ and assume $U_1 \subset U_2$. This open immersion is given by a morphism i induced by a ring homomorphism $A_2 \xrightarrow{\phi} A_1$. Let $g \in A_2$ such that $D(g) \subset U_1$. Then I claim that i induces an isomorphism of $D(\phi(g))$ with $D(g)$. In fact, since there is a unique open subscheme structure on an open subspace, it is enough to show that i is a homeomorphism, and for this it is enough to show it is bijective (since it is open and continuous). Namely, it is enough to show that as sets, $i^{-1}(Spec(A_{2,g})) = Spec\left(A_{1,\phi(g)}\right)$. I will supress notation for the localization homomorphisms and induced open immersions.

Let $p \in i^{-1}(Spec(A_{2,g}))$, so $g \notin \phi^{-1}(p) = q$. If $\phi(g) \in p$, then $g \in \phi^{-1}(p)$, so $\phi(g) \notin p$, and $p \in Spec\left(A_{1,\phi(g)}\right)$.

Now let $p \in Spec\left(A_{1,\phi(g)}\right)$, so $\phi(g) \notin p$. Thus $g \notin i(p) = \phi^{-1}(p)$, so $i(p) \in Spec(A_{2,g})$. This implies $p \in i^{-1}(i(p)) \subset i^{-1}(Spec(A_{2,g}))$, which proves the claim, and the theorem. □