MATH 320 PRACTICE PROBLEMS FOR THE FINAL EXAM

Here are some practice problems for the final exam. During the final, you may use the textbook and your class notes but no other materials. The final will have 7 problems. You must be comfortable finding solutions of linear systems of equations, finding bases and calculating dimensions of vector spaces, calculating the image, kernel, rank, nullity of homomorphisms, finding an orthonormal basis of a vector space using the Gram-Schmidt algorithm, finding orthogonal projections, calculating Fourier coefficients, calculating determinants of matrices, finding the characteristic polynomial, minimal polynomial, eigenvalues and eigenvectors of matrices, diagonalizing matrices, finding the Jordan canonical form of a matrix. You will be tested on all these basic skills that you should have acquired in this course. You will also be expected to prove basic statements on topics covered in the course.

Problem 1. Find all the solutions of the system of linear equations

$$\begin{aligned} x - 2y + 3z - w &= 2 \\ x + 5y - z + 3w &= -2 \\ x + y + 4z - w &= 1 \end{aligned}$$

Problem 2. Find a linear system of equations that has as solution set

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + c_1 \begin{pmatrix} -1 \\ 2 \\ -3 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 2 \\ -1 \\ -1 \end{pmatrix}$$

Problem 3. Let A be the following matrix.

$$A = \begin{pmatrix} 2 & 4 & 1 & 1 \\ -2 & 1 & 2 & 1 \\ 2 & 9 & 4 & 3 \end{pmatrix}$$

(1) Find a basis of the kernel of A. Determine the nullity of A.
(2) Apply the Gram-Schmidt algorithm to your basis in the previous part to find an orthonormal basis (for the usual inner product on $\mathbb{R}^4$) of the kernel of A.
(3) Find a basis of the image of A. Determine the rank of A.
(4) Apply the Gram-Schmidt algorithm to your basis in the previous part to find an orthonormal basis (for the usual inner product on $\mathbb{R}^3$) of the image of A.

Problem 4. Let $M_{2\times 2}$ denote the space of 2×2 matrices with real coefficients. Show that

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = a_1a_2 + 2b_1b_2 + c_1c_2 + 2d_1d_2$$

defines an inner product on $M_{2\times 2}$. Find an orthogonal basis of the subspace

$$S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + 3b - c = 0 \right\}$$

of $M_{2\times 2}$ defined by with respect to this inner product.

Problem 5. Find an orthonormal basis (for the usual inner product) of the subspace of $\mathbb{R}^6$ defined by

$$S = \{(x_1, \ldots, x_6) \mid x_1 + 3x_2 - x_3 = x_4 - 2x_5 + x_6 = 0\}.$$

Problem 6. Calculate the determinants of the following matrices. Determine whether the matrices are singular or non-singular.

$$A = \begin{pmatrix} 2 & 4 & 1 & 1 \\ -2 & 1 & 2 & 1 \\ 2 & 9 & 4 & 3 \\ 2 & 1 & 2 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 4 & 1 \\ -2 & 2 & 1 \\ 9 & 4 & 3 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 4 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Problem 7. Calculate the characteristic polynomials of the following matrices. For each of the matrices determine the eigenvalues and eigenvectors. Decide whether the matrix is diagonalizable. If the matrix is diagonalizable, find a matrix that conjugates it to a diagonal matrix. If not, find the Jordan normal form of the matrix. Calculate A^{100}.

$$A = \begin{pmatrix} 2 & 4 \\ -2 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 3 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 4 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$A = \begin{pmatrix} -5 & -8 \\ \frac{9}{2} & 7 \end{pmatrix}$$

Problem 8. Suppose that the characteristic polynomial of a matrix A is $(\lambda - 1)^3(\lambda - 2)^2$. Determine all the possible Jordan normal forms of A up to conjugation.

Problem 9. Suppose the characteristic polynomial of a matrix A is $\lambda^3(\lambda - 1)(\lambda - 2)$. If the nullity of A is two, what are the possible Jordan normal forms of A up to conjugation? What if the characteristic polynomial were $\lambda^4(\lambda - 1)(\lambda - 2)$?

Problem 10. Determine whether the following statements are TRUE or FALSE. If the statement is true, give a proof. If the statement is false, provide a counterexample. You will receive no credit for simply saying TRUE/FALSE

(1) Let A and B be $n \times n$ matrices such that $AB = BA$. If v is an eigenvector of A, then Bv is also an eigenvector of A with the same eigenvalue.
(2) Recall that an orthogonal matrix is an $n \times n$ matrix such that $AA^T = I_n$. If λ is a root of the characteristic polynomial of an orthogonal matrix A, then $1/\lambda$ is also a root of the characteristic polynomial of A.
(3) Recall that a special orthogonal matrix is an $n \times n$ matrix such that $AA^T = I_n$ and $\det(A) = 1$. A special orthogonal matrix has a fixed vector (i.e., a vector v such that $Av = v$).
(4) The eigenvalues of A and A^T are equal.
(5) The eigenvectors of A and A^T are equal.
(6) The eigenvalues of A and A^2 are equal.
(7) The eigenvectors of A and A^2 are equal.
(8) The dimension of the zero eigenspace of a matrix A is equal to the nullity of A.
(9) Let $[A, B] = AB - BA$ be the commutator of two $n \times n$ matrices. The commutator satisfies the Jacobi identity
$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$$

Problem 11. Calculate the Fourier coefficients of the function
$$f(x) = \begin{cases} 1 & \text{if } (2k-1)\,\pi < x \leq 2k\,\pi \\ 0 & \text{if } 2k\,\pi < x \leq (2k+1)\,\pi \end{cases}$$

Problem 12. Determine whether the following matrix
$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 4 \end{pmatrix}$$
is diagonalizable. If so, find a matrix that conjugates A to a diagonal matrix. Calculate A^{100}.

Problem 13. Find the Jordan normal form of the matrix
$$A = \begin{pmatrix} 3 & 1 & 0 \\ -1 & 1 & 1 \\ -16 & -7 & 1 \end{pmatrix}$$
Calculate A^{100}.