HYPERBOLIC VOLUME AND MOD p HOMOLOGY

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Abstract. If M is a closed, orientable hyperbolic 3-manifold such that $\dim_{\mathbf{Z}_p} H_1(M; \mathbf{Z}_p) \geq 5$ for some prime p, then M contains a hyperbolic ball of radius $(\log 5)/4$. There is also a related result in higher dimensions.

Introduction

In [**8**, Proposition 5.4] it was shown that if M is an orientable hyperbolic 3-manifold, and if for some prime p the $\mathbf{Z}_p$-vector space $H_1(M, \mathbf{Z}_p)$ has dimension at least 4, then M contains a ball of radius $(\log 3)/4$. This implies that the volume of M is greater than 0.11. In this paper we shall prove:

Theorem A. *Let M be a closed, orientable hyperbolic 3-manifold. Suppose that for some prime p, the dimension of the $\mathbf{Z}_p$-vector space $H_1(M; \mathbf{Z}_p)$ is at least 5. Then M contains a hyperbolic ball of radius* $(\log 5)/4$. *In particular, the volume of M is greater than* 0.35.

By a *hyperbolic ball* in a hyperbolic n-manifold M we mean an open subset of M which is path-isometric to an open ball in hyperbolic n-space $\mathbf{H}^n$. The volume estimate in the theorem can be deduced from the existence of a hyperbolic ball of radius $(\log 5)/4$ by using density estimates for sphere-packings as in [**7**] (see also [**3**]).

Theorem 6.1 of this paper asserts that the conclusion of Theorem A remains true under the hypothesis that any three elements of $\pi_1(M)$ generate an infinite-index subgroup of $\pi_1(M)$. The latter hypothesis is actually weaker than that of Theorem A; this is because, according to [**8**, Proposition 1.1], if k is a positive integer and M is a closed, orientable 3-manifold such that $\dim_{\mathbf{Z}_p} H_1(M; \mathbf{Z}_p) \geq k + 2$ for some prime p, then any k elements of $\pi_1(M)$ generate an infinite-index subgroup of $\pi_1(M)$. Thus Theorem A is in fact a special case of Theorem 6.1.

We will also prove a related result in higher dimensions. Recall that the *rank* of a finitely generated group F is defined to be the minimal cardinality of a generating set for F. A

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group Γ is said to be k-free, where k is a non-negative integer, if every subgroup of Γ with rank at most k is free. We have:

Theorem B. *Let M be a closed hyperbolic manifold of dimension $n \geq 3$. Suppose that $\pi_1(M)$ is 3-free. Then M contains a hyperbolic ball of radius*

$$\frac{\log 5}{2(n-1)}.$$

The first five sections of the paper are devoted to the proof of Theorem B. One regards the hyperbolic n-manifold M as a quotient $\mathbf{H}^n/\Gamma$, where Γ is a discrete, torsion-free group of isometries of $\mathbf{H}^n$. For each maximal cyclic subgroup X of Γ and each $\lambda > 0$ one considers the set $Z_\lambda(X)$ consisting of all points of $\mathbf{H}^n$ that are moved a distance less than λ by some non-trivial element of X. It is an elementary observation (Proposition 3.2) that if M contains no hyperbolic ball of radius $\lambda/2$ then the non-empty sets of the form $Z_\lambda(X)$ constitute an open covering of H^n. The nerve of this covering is a simplicial complex K. The geometric properties of the sets in the covering—which are fairly well-behaved neighborhoods of the axes of the corresponding cyclic subgroups—impose topological restrictions on K: it is connected, and the link of every vertex is connected.

As the sets in the covering are determined by certain maximal cyclic subgroups of Γ, the vertices of K have a natural labeling by maximal cyclic subgroups. As Γ is 3-free, the vertices of any 2-simplex of K generate a free group. However, if $\lambda = (\log 5)/(n-1)$, the discreteness of Γ can be used to show that this free group is never of rank 3: this depends on Proposition 3.5, which is an elementary geometric argument based on ideas that appeared in [**4**] and [**8**]. Thus in the labeling of the vertices of K by cyclic groups, the three cyclic groups labeling the vertices of any given 2-simplex generate a free group of rank 2. Using the topological properties of K and elementary facts about free groups, one can conclude that the group generated by all the labeling cyclic groups—i.e. by all cyclic groups X for which $Z_\lambda(X) \neq \emptyset$—is locally a free group of rank 2. By pushing the group theory a bit further one can then deduce that Γ is itself a free group of rank 2, and this is impossible as Γ is the fundamental group of a closed aspherical manifold.

In Section 1 we prove some elementary properties of the sets $Z_\lambda(X)$, for X any cyclic group of loxodromic isometries of $\mathbf{H}^n$. In Section 2 we prove a purely topological result about nerves of coverings of topological spaces. In Section 3 the results of the two preceding sections are combined to establish the relevant topological properties of the complex K. The proof of the geometric result alluded to above, Proposition 3.5, is also given. In Section 4 we establish the relevant facts about free groups and labeled complexes of groups. In Section 5 the results of Sections 3 and 4 are combined to give the proof of Theorem B.

Actually this is all done in a somewhat more refined setting, and gives a result, Theorem 5.1, which is more technical than Theorem B but includes it as a special case. In Section 6 we will combine Theorem 5.1 with the specifically 3-dimensional results of [**3**] to deduce Theorem 6.1 and hence Theorem A.

The following conventions will be used throughout the paper. The hyperbolic distance in $\mathbf{H}^n$ will be denoted dist. If S is a subset of $\mathbf{H}^n$ and r is a positive number, $\mathrm{nbhd}_r(P)$ will denote the open r-neighborhood of S, i.e. the set of all points whose minimum distance from S is strictly less than r.

If σ is a simplex in the simplicial complex K, the *link* of σ in K, denoted by $\mathrm{link}_K(\sigma)$, consists of all simplices τ such that (i) $\sigma \cap \tau = \emptyset$ and (ii) σ and τ span a simplex of K. The *support* of a simplex σ in K is the subcomplex of K consisting of σ and all its faces; it will be denoted by $|\sigma|$.

If S is a subset of a group Γ, we denote by $\langle S \rangle$ the subgroup of Γ generated by S. (If $S = \{x_1, \ldots, x_r\}$, we may also write $\langle x_1, \ldots, x_r \rangle$ for $\langle S \rangle$.)

Let $z_1, \ldots, z_r$ be elements of a group Γ. We shall say that $z_1, \ldots, z_r$ are *independent* if they freely generate a (free, rank-r) subgroup of Γ. (Here we regard $\{z_1, \ldots, z_r\}$ as an indexed r-tuple; in particular, if two of the z_i coincide, then $z_1, \ldots, z_r$ are not independent.)

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Section 1. Loxodromic isometries and displacement cylinders

1.1. Recall that an isometry x of $\mathbf{H}^n$ is *loxodromic* if there is an x-invariant line $A(x)$ in $\mathbf{H}^n$, and x acts on $A(x)$ as a translation through some distance $\mathrm{length}\, x > 0$. The line $A(x)$ is unique, and is called the *axis* of x.

If x is an isometry of $\mathbf{H}^n$ we define a continuous non-negative-valued function D_x on $\mathbf{H}^n$ by $D_x(P) = \mathrm{dist}(P, x \cdot P)$. Note that $D_x = D_{x^{-1}}$.

1.2. Suppose that x is loxodromic with length l. Let P be any point of $\mathbf{H}^n - A(x)$, and let Q denote the point of $A(x)$ closest to P. Set $P' = x \cdot P$ and $Q' = x \cdot Q$. We have $\angle PQQ' = \angle P'Q'Q = \pi/2$, $\mathrm{dist}(Q, Q') = l$, and $\mathrm{dist}(P, Q) = \mathrm{dist}(P', Q') = r$, where $r = r_x(P)$ denotes the perpendicular distance from P to $A(x)$. Let $\theta = \theta_x(P)$ denote the dihedral angle between the planes PQQ' and $P'Q'Q$. Setting $D = D_x(P) = \mathrm{dist}(P, P')$, one sees by elementary hyperbolic geometry that

$$\cosh \mathrm{dist}(P, P') = \cosh l + (\sinh^2 r)(\cosh l - \cos\theta),$$

i.e.

$$\cosh D_x(P) = \cosh l + (\sinh^2 r_x(P))(\cosh l - \cos\theta_x(P)). \tag{1.2.1}$$

This formula is clearly valid for $P \in A(x)$ if we assign an arbitrary value to $\theta_x(P)$. In particular we recover the familiar fact that $\mathrm{dist}(P, x \cdot P) \geq l$ for every point $P \in \mathbf{H}^n$, with equality if and only if $P \in A(x)$.

1.3. Note that θ_x is constant on every ray which is perpendicular to $A(x)$ and has its endpoint in A. If ρ is such a ray, the function $r_x|\rho$ maps ρ homeomorphically onto $[0, \infty)$ and thus defines a coordinate r on ρ. It follows from (1.2.1) that $D_x|\rho$ is a strictly monotonically increasing function of r and goes to infinity with r.

(In the case where $n = 3$ and x preserves orientation, θ_x is constant on all of H^3. This fact will not be used in the present paper.)

1.4. Now for any loxodromic isometry x of $\mathbf{H}^n$ we define a non-negative-valued function E_x on $\mathbf{H}^n$ by setting $E_x(P) = \min_{d\geq 1} D_{x^d}(P)$ for every $P \in \mathbf{H}^n$.

Proposition. *Let x be a loxodromic isometry of $\mathbf{H}^n$. Then E_x is continuous, and* $\min_{P\in\mathbf{H}^n} E_x(P) =$ $\operatorname{length} x$*. Furthermore, for every $C > 0$ there exists $R > 0$ such that $E_x(P) > C$ for every point P such that $r_x(P) > R$.*

Proof. Set $l = \operatorname{length} x$. Then $\operatorname{length} x^d = dl$ for every integer $d > 0$. Hence for each d we have $\min_{P\in\mathbf{H}^n} D_{x^d}(P) = dl$, and so

$$\min_{P\in\mathbf{H}^n} E_x(P) = \min_{d>0} dl = l.$$

To show that E_x is continuous on $\mathbf{H}^n$ it suffices to show that it is continuous on the set $H_\alpha = D_x^{-1}([0,\alpha))$ for each $\alpha > 0$. For any integer $d > \frac{\alpha}{l}$ we have $D_{x^d}(P) \geq dl > \alpha$ for every $P \in \mathbf{H}^n$. It follows that for any $p \in H_\alpha$ we have $E_x(P) = \min_{1\leq d\leq[\frac{\alpha}{l}]} D_{x^d}(P)$; since each of the functions $D_{x^d}(P)$ is continuous on $\mathbf{H}^n$, it follows that E_x is continuous on H_α.

Now let C be any positive constant, and let R be a constant such that $(\sinh^2 R)(\cosh l - 1) > \cosh C$. Since x^d has translation length dl, it follows from (1.2.1) that $D_{x^d}(P) > C$ for every positive integer $d > 0$ and for every P with $r_x(P) \geq R$. Hence $E_{x^d}(P) > C$ whenever $r_x(P) \geq R$. □

1.5. Proposition. *Let x be a loxodromic isometry of $\mathbf{H}^n$. Let ρ be any ray in $\mathbf{H}^n$ which has its endpoint in $A(x)$ and is perpendicular to $A(x)$; let us identify ρ isometrically with $[0,\infty)$. Then $f = E_x|\rho$ is monotonically increasing, and $f(P)$ tends to infinity with P.*

Proof. According to 1.3, the function $D_{x^d}|\rho$ is strictly monotonically increasing for every $d > 0$. Hence $f = E_x|\rho = \min_{d\geq 1} D_{x^d}|\rho$ is also strictly monotonically increasing. The final assertion of Proposition 1.4 implies that $f(P)$ tends to ∞ with P. □

For any loxodromic x and any $\lambda > 0$ we set $Z_\lambda(x) = E_x^{-1}[0,\lambda)$.

1.6. Proposition. *Let x be a loxodromic isometry of $\mathbf{H}^n$. For any $\lambda \leq \operatorname{length} x$ we have $Z_\lambda(x) = \emptyset$. For any $\lambda > \operatorname{length} x$ the set $Z_\lambda(x)$ is an open contractible neighborhood of $A(x)$ and is contained in $\operatorname{nbhd}_R A(x)$ for some constant $R > 0$. Furthermore, the frontier of $Z_\lambda(x)$ in $\mathbf{H}^n$ is the set $Q_\lambda(x) = E_x^{-1}(\{\lambda\})$, and $Q_\lambda(x)$ is homeomorphic to $S^{n-2} \times \mathbf{R}$.*

Proof. Set $l = \operatorname{length} x$. Since $\min_{P\in\mathbf{H}^n} E_x(P) = l$ by Proposition 1.4, we have $Z_\lambda(x) = \emptyset$ for any $\lambda \leq l$. On the other hand, since D_x is identically equal to l on $A(x)$, we have $A(x) \subset Z_\lambda(x)$ for any $\lambda > l$; in view of the continuity of E_x it follows that $Z_\lambda(x)$ is an open neighborhood of $A(x)$. To show that $Z_\lambda(x)$ is contractible in this case, we consider any ray $\rho \subset \mathbf{H}^n$ which has its endpoint in $A(x)$ and is perpendicular to $A(x)$. It follows from Proposition 1.5 that $Z_\lambda(x) \cap \rho$ is a half-open line segment with the same endpoint as ρ; since this holds for every such ray ρ, the contractibility of $Z_\lambda(x)$ is clear.

It follows immediately from Proposition 1.4 that $Z_\lambda(x) \subset \operatorname{nbhd}_R A(x)$ for some constant $R > 0$.

The continuity of E_x implies that the frontier of $Z_\lambda(x)$ in $\mathbf{H}^n$ is contained in the set $Q_\lambda(x) = E_x^{-1}(\{\lambda\})$. To prove the reverse inclusion, we consider any point $P \in Q_\lambda(x)$, and we let ρ_P denote the unique ray in $\mathbf{H}^n$ which has its endpoint in A, is perpendicular to $A(x)$ and contains P. According to 1.5, the function $f = E_x|\rho_P$ is strictly monotonically increasing. Since $f(P) = \lambda$, the monotonicity of f implies that P lies in the frontier relative to ρ_P of the set $f^{-1}[0, \lambda) = Z_\lambda(x) \cap \rho_P$. In particular, P lies in the frontier of $Z_\lambda(x)$ in $\mathbf{H}^n$.

It remains to show that $Q_\lambda(x)$ is homeomorphic to $S^{n-2} \times \mathbf{R}$. For this purpose we consider the set $Q^*(x) \subset \mathbf{H}^n$ consisting of all points whose perpendicular distance from $A(x)$ is 1. If P is any point of $Q_\lambda(x)$, and ρ_P is defined as above, then $\rho_P \cap Q^*(x)$ consists of a single point which we denote $h(P)$. This defines a continuous map $h : Q_\lambda(x) \to Q^*(x)$. It follows immediately from Proposition 1.5 that h is a bijection. On the other hand, since we have shown that $Z_\lambda(x) \subset \operatorname{nbhd}_R A(x)$ for some constant $R > 0$, it is clear that every compact subset of $Q^*(x)$ has bounded pre-image under h. But $Q_\lambda(x)$ is closed in $\mathbf{H}^n$ since it is the frontier of $Z_\lambda(x)$. Thus $Q_\lambda(x)$ is locally compact and h is a proper map. It follows that h is a homeomorphism. Since $Q^*(x)$ is clearly homeomorphic to $S^{n-2} \times \mathbf{R}$, this completes the proof. □

We remark that in the case that $n = 3$ and x preserves orientation we have $Z_\lambda(x) = \operatorname{nbhd}_R A(x)$ for some $R > 0$. This fact will not be used in the present paper.

1.7. For any loxodromic isometry x of $\mathbf{H}^n$, it follows from 1.1 that $D_{x^d} = D_{x^{-d}}$ for every $d > 0$. This implies that $E_x = E_{x^{-1}}$, and hence that $Z_\lambda(x) = Z_\lambda(x^{-1})$ for every $\lambda > 0$. Hence if X is any infinite cyclic subgroup of Γ with a loxodromic generator x we may unambiguously write $Z_\lambda(X) = Z_\lambda(x)$ for any positive number λ. This notation will be used extensively in the next two sections.

Section 2. Nerves and connectedness

2.1. By an *open covering* of a topological space H we shall mean an indexed family $(U_i)_{i \in I}$ of non-empty open sets in H such that $\bigcup_{i \in I} U_i = H$. Note that we may have $U_i = U_j$ for distinct indices i and j. The *nerve* of the covering $(U_i)_{i \in I}$ is an abstract simplicial complex with an indexed vertex set $(v_i)_{i \in I}$, where $v_i = v_j$ if and only if $i = j$. A collection $\{v_{i_0}, \ldots, v_{i_k}\}$ of vertices, where $i_0, \ldots, i_k$ are distinct indices in I, spans a k-simplex if and only if $U_{i_0} \cap \cdots \cap U_{i_k} \neq \emptyset$.

If H is connected, any open covering of H has connected nerve. (This depends on our requirement that the sets in an open covering be non-empty.)

2.2. Proposition. *Let $(U_i)_{i \in I}$ be a covering of a topological space H. Suppose that*
(i) for every $i \in I$ the set U_i is connected and has connected frontier, and
(ii) for any two distinct indices $i, j \in I$ we have $U_i \not\subset U_j$.
Then the link of every vertex in the nerve of $(U_i)_{i \in I}$ is connected.

Proof. Let K denote the nerve of $(U_i)_{i \in I}$. Suppose that we are given a vertex of K, say v_s for some $s \in I$. Set $C = \operatorname{link}_K v_s$. We are required to show that C is a connected simplicial complex. Let us write the set of vertices of C as an indexed set $(v_i)_{i\in J}$, where J is a subset of I. We have $j \in J$ if and only if v_s and v_j span a 1-simplex of K; by the definition of the nerve, this is equivalent to saying that $j \neq s$ and $U_j \cap U_s \neq \emptyset$.

We denote by Q the frontier of U_s in H. By hypothesis (i), Q is a connected space. Consider the indexed family $(U_j \cap Q)_{j\in J}$ of open sets in Q. We claim that this family is an open covering of Q.

First we must show that $U_j \cap Q \neq \emptyset$ for any $j \in J$. Since for $j \in I$ we have $j \neq s$, hypothesis (ii) implies that $U_j \not\subset U_s$. Since U_j is connected by hypothesis (i), and since $U_j \cap U_s \neq \emptyset$ when $j \in J$, it follows that U_j meets the frontier Q of U_s as required. Next we must show that $Q = \bigcup_{j\in J}(Q \cap U_j)$, i.e. that $Q \subset \bigcup_{j\in J}(U_j)$. Given any point $q \in Q$, we have $q \in U_j$ for some $j \in I$; since $q \notin U_s$ we have $s \neq j$. But since $q \in \overline{U}_s$ we must have $U_j \cap U_s \neq \emptyset$. This shows that $j \in J$, and completes the proof that $(U_j \cap Q)_{j\in J}$ is an open covering of Q.

Let E denote the nerve of the covering $(U_j \cap Q)_{j\in J}$. Let w_j denote the vertex of E corresponding to the index $j \in J$. If $w_{j_0}, \ldots, w_{j_k}$ span a k-simplex of E then $(Q \cap U_{j_0}) \cap \cdots \cap (Q \cap U_{j_k}) \neq \emptyset$; hence in particular, $(U_{j_0} \cap \cdots \cap U_{j_k}) \cap U_s \neq \emptyset$, so that $v_{j_0}, \ldots, v_{j_k}, v_s$ span a $(k+1)$-simplex of K. This means that $v_{j_0}, \ldots, v_{j_k}$ span a k-simplex of C. This shows that E is simplicially isomorphic to a subcomplex of C containing all the vertices of C.

Since Q is connected, the nerve E of the covering $(U_j \cap Q)_{j\in J}$ is connected. Thus there is a connected subcomplex of C containing all the vertices of C. It follows that C is itself connected. □

Section 3. Discrete groups and coverings of hyperbolic space

3.1. In this section, M will denote a closed hyperbolic manifold of some dimension $n \geq 2$. We may regard M as the quotient of $\mathbf{H}^n$ by a co-compact, discrete, torsion-free group Γ of isometries. We recall some elementary properties of Γ. Since Γ is co-compact, each non-trivial element x of Γ is loxodromic. The centralizer $C(x)$ of x is cyclic and consists of all elements having the same axis as x. In particular $C(x)$ is the unique maximal cyclic subgroup containing x. For two non-trivial elements x and y of Γ we have $C(x) = C(y)$ if and only if x and y commute, or equivalently if and only if $A(x) = A(y)$. Thus there is a natural one-one correspondence between maximal cyclic subgroups of Γ and axes of elements of Γ.

3.2. Proposition. *Suppose that λ is a positive number such that M contains no hyperbolic ball of radius $\frac{\lambda}{2}$. Then we have*

$$\mathbf{H}^n = \bigcup_X Z_\lambda(X),$$

where X ranges over all maximal cyclic subgroups of Γ.

Proof. Let P be any point of $\mathbf{H}^n$. The hypothesis that M contains no hyperbolic ball of radius $\frac{\lambda}{2}$ implies that $\mathrm{dist}(P, x_0 \cdot P) < \lambda$ for some $x_0 \in \Gamma - 1$. (Indeed, if $\mathrm{dist}(P, x \cdot P) \geq \lambda$

for every $x \in \Gamma - \{1\}$ then by the triangle inequality $B = \text{nbhd}_{\lambda/2}(P)$ is disjoint from $x \cdot B = \text{nbhd}_{\lambda/2}(x \cdot P)$ for every $x \in \Gamma - \{1\}$; hence the covering projection maps B injectively into M, and its image is a hyperbolic ball of radius $\frac{\lambda}{2}$.) Now by 3.1, $X_1 = C(x_0)$ is a maximal cyclic subgroup of Γ, and x_0 is a positive power of some generator x_1 of X_1. By the definitions we have $E_{x_1}(P) \leq D_{x_0}(P) = \text{dist}(P, x_0 \cdot P) < \lambda$, so that $P \in Z_{x_1}(P) = Z_{X_1}(P)$. Since $P \in \mathbf{H}^n$ was arbitrary, the conclusion of the lemma follows. □

3.3. Proposition. *Suppose that X and X' are maximal cyclic subgroups of Γ, and suppose that for some $\lambda > 0$ we have $\emptyset \neq Z_\lambda(X) \subset Z_\lambda(X')$. Then $X = X'$.*

Proof. Let x and x' be generators of X and X' respectively, and set $A = A_x$, $A' = A_{x'}$. By Proposition 1.6 we have $Z_\lambda(X') \subset \text{nbhd}_R A'$ for some $R > 0$. Hence $Z_\lambda(X) \subset \text{nbhd}_R A'$. Now let $\overline{\mathbf{H}}^n$ denote the union of $\mathbf{H}^n$ with the sphere at infinity S^{n-1}_∞. We give $\overline{\mathbf{H}}^n$ the natural topology, in which it is homeomorphic to a closed n-ball. Let $\overline{Z}$ denote the closure of $Z_\lambda(X)$ in $\overline{\mathbf{H}}^n$. Since $Z_\lambda(X) \subset \text{nbhd}_R A'$, we have $\overline{Z} \cap S^{n-1}_\infty \subset A' \cap S^{n-1}_\infty = \{P, Q\}$, where P and Q are the fixed points of x' in S^{n-1}_∞. Thus $\{P, Q\}$ is invariant under x. Hence x^2 fixes P and Q. Since x^2 is loxodromic with axis A it follows that P and Q are the endpoints of A and hence that $A = A'$. By 3.1 this implies $X = X'$. □

3.4. Suppose that λ is a positive number such that M contains no hyperbolic ball of radius $\lambda/2$. Let $\mathcal{X} = \mathcal{X}_\lambda(M)$ denote the set of all maximal cyclic subgroups X of Γ such that $Z_\lambda(X) \neq \emptyset$. Proposition 3.2 implies that the indexed family $(Z_\lambda(X))_{X \in \mathcal{X}}$ is an open covering of $\mathbf{H}^n$ (see 2.1). We will denote the nerve of this covering by $K_\lambda(M)$.

Proposition. *Let λ be a positive number such that M contains no hyperbolic ball of radius $\lambda/2$. Then $K = K_\lambda(M)$ is a connected complex with more than one vertex, and the link of every vertex of K is connected.*

Proof. Since $\mathbf{H}^n$ is connected, the nerve of any open covering of $\mathbf{H}^n$ is connected. Set $\mathcal{X} = \mathcal{X}_\lambda(M)$ and $K = K_\lambda(M)$. If K had only one vertex, we would have $\mathbf{H}^n = Z_\lambda(X)$ for some $X \in \mathcal{X}$. This is impossible since by Proposition 1.6 we have $Z_\lambda(X) \subset \text{nbhd}_R(A)$ where A is the axis of a generator of X and R is some positive number. To show that the link of every vertex of K is connected we apply Proposition 2.2. According to Proposition 1.6, for each $X \in \mathcal{X}$ the set $Z_\lambda(X)$ is contractible and hence connected, and its frontier is homeomorphic to $S^{n-2} \times \mathbf{R}$ and is therefore connected since $n \geq 3$. Thus hypothesis (i) of Proposition 2.2 holds. That hypothesis (ii) holds is precisely the content of Proposition 3.3. □

3.5. As we explained in the introduction, the following result is the basic geometric fact underlying the proofs of Theorems A and B. The proof is a slight variant of the proof of [**8**, Proposition 5.2 and Corollary 5.3]; see also [**4**].

Proposition. *Let $x_1, \ldots, x_r$ be independent elements of Γ. Set*

$$\lambda = \frac{\log(2r-1)}{n-1}.$$

Then $Z_\lambda(x_1) \cap \cdots \cap Z_\lambda(x_r) = \emptyset$.

Proof. Suppose that P is a point of $Z_\lambda(x_1) \cap \cdots \cap Z_\lambda(x_r)$. For each $i \in \{1, \ldots, r\}$ we have $E_{x_i}(P) < \lambda$, and hence $D_{y_i}(P) < \lambda$ for some positive power $y_i = x_i^{d_i}$ of x_i. Clearly $y_1, \ldots, y_r$ are independent. We fix a number $\lambda' < \lambda$ such that $\operatorname{dist}(P, y_i \cdot P) = D_{y_i}(P) < \lambda'$ for $i = 1, \ldots, r$. It then follows by induction on $m \geq 1$, using the triangle inequality and the fact that the y_i are isometries, that if $\gamma \in \Gamma$ is given by a word of length m in $y_1, \ldots, y_r$ then $\operatorname{dist}(P, \gamma \cdot P) < m\lambda'$.

For each $m \geq 1$, let S_m denote the set of all elements of Γ that are expressible as reduced words of length m in $y_1, \ldots, y_r$. Since $y_1, \ldots, y_r$ are independent, S_m has cardinality exactly $(2r)(2r-1)^{m-1}$. Let b be an open ball about P such that $\gamma \cdot b \cap b = \emptyset$ for every $\gamma \in \Gamma - \{1\}$. Let ρ denote the radius of b, and v its volume. Then the balls $\gamma \cdot b$ for $\gamma \in S_m$ are pairwise disjoint and are contained in $\operatorname{nbhd}_{m\lambda'+\rho}(P)$. Hence

$$(2r)(2r-1)^{m-1} v \leq \operatorname{vol} \operatorname{nbhd}_{m\lambda'+\rho}(P) < C \exp{(n-1)(m\lambda' + \rho)},$$

where C is a constant depending only on the dimension n. Hence

$$(2r-1)^m < C' \exp{(n-1)m\lambda'},$$

where C' is a constant depending on n and ρ but independent of m. If in the last inequality we take logarithms of both sides, divide by m and take limits as $m \to \infty$, we obtain $\log(2r-1) \leq (n-1)\lambda'$, which is impossible since

$$\lambda' < \lambda = \frac{\log(2r-1)}{n-1}.$$

□

Section 4. Structure of 3-free groups

4.1. Let W be a subgroup of a group Γ, and let k be positive integer. We shall say that Γ is *k-free over W* if every subgroup of Γ which contains W and has rank $\leq k$ is free (of some rank $\leq k$). Note that a group is k-free if and only if it is k-free over the trivial subgroup; and that a k-free group is k-free over every subgroup.

A group Γ will be said to have *local rank* $\leq k$, where k is a positive integer, if every finitely generated subgroup of Γ is contained in a subgroup of rank $\leq k$. The local rank is the smallest integer k with this property, and is defined to be ∞ if no such integer exists. Note that for a finitely generated group, the local rank is equal to the rank.

The following result plays the role of an induction step in the proofs of the two main results of this section, Propositions 4.3 and 4.4.

4.2. Lemma. *Let x, y and z be elements of a group Γ. Suppose that x and y do not commute, and that x, y and z are not independent. Let A be a subgroup of Γ which contains*

x and y and has local rank 2. Suppose that Γ is 3-free over some finitely generated subgroup J of A. Then the group $\langle A \cup \{z\}\rangle$ is also of local rank 2.

Proof. Since $\langle A \cup \{z\}\rangle$ contains the non-commuting elements x and y, it must have local rank > 1. We must show that it has local rank ≤ 2.

Set $B = \langle A \cup \{z\}\rangle$. Let B_0 be any finitely generated subgroup of B. Then there is a finitely generated subgroup A_1 of A such that B_0 is contained in the subgroup $B_1 = \langle \{z\} \cup A_1\rangle$. After possibly replacing A_1 by a larger finitely generated subgroup we may assume that $x, y \in A_1$ and that $J \leq A_1$. Since A has local rank ≤ 2, we may assume after further enlarging A_1 that A_1 has rank ≤ 2. Hence B_1 has rank at most 3. Since $J \leq B_1$, and since Γ is 3-free over J, it follows that B_1 is free of some rank at most 3. We claim that B_1 cannot have rank 3. This will imply that $B_1 \geq B_0$ has rank ≤ 2, and will complete the proof that B has local rank ≤ 2.

Assume that B_1 is free of rank 3. Since A_1 has rank ≤ 2 it is generated by two elements u and v. Then u, v and z generate the rank-3 free group B_1. Hence by [**6**, p. 59], B_1 is freely generated by these three elements. Thus we may regard B_1 as a free product $A_1 * \langle z\rangle$. Now $T = \langle x, y\rangle \leq B_1$ is free by the Nielsen-Schreier theorem, and has rank ≤ 2; since x and y do not commute, T must be free of rank exactly 2, and must therefore be free on x and y. But since B_1 has been identified with a free product $A_1 * \langle z\rangle$, the subgroup $\langle x, y, z\rangle = \langle T \cup \{z\}\rangle$ is identified with a free product $T * \langle z\rangle$, and is therefore freely generated by x, y and z. This is a contradiction since x, y and z are not independent. ☐

4.3. Let Γ be a group. By a Γ-*labeled complex* we shall mean an ordered pair $(K, (X_v)_v)$, where K is a simplicial complex and $(X_v)_v$ is a family of cyclic subgroups of Γ indexed by the vertices of K. If $(K, (X_v)_v)$ is a Γ-labeled complex then for any subcomplex L of K we shall denote by $\Theta(L)$ the subgroup of Γ generated by all the groups X_v, where v ranges over the vertices of L.

Proposition. *Let $(K, (X_v)_v)$ be a Γ-labeled complex. Suppose that K is connected and has more than one vertex, and that the link of every vertex of K is connected. Suppose that for every 1-simplex e of K the group $\Theta(|e|)$ is non-abelian and Γ is 3-free over $\Theta(|e|)$. Suppose also that there is no 2-simplex σ of K such that $\Theta(|\sigma|)$ is free of rank 3. Then $\Theta(K)$ has local rank 2.*

Proof. We shall say that a subcomplex L of K is *good* if (i) L is connected and contains more than one vertex, (ii) $\Theta(L)$ has local rank 2, and (iii) Γ is 3-free over some finitely generated subgroup of $\Theta(L)$.

If e is any 1-simplex of K, then $\Theta(|e|)$ is by definition generated by two elements, and the hypothesis of the lemma implies that $\Theta(|e|)$ is non-abelian; in particular $\Theta(|e|)$ has local rank 2. The hypothesis also implies that Γ is 3-free over $\Theta(|e|)$. Thus $|e|$ is a good subcomplex of K. It now follows from Zorn's Lemma that there exists a maximal good subcomplex L_0 of K. We shall complete the proof by showing that $L_0 = K$.

4.3.1. Claim. *The complex L_0 is a full subcomplex of K. (This means that any simplex whose vertices lie in L_0 is itself a simplex of L_0.)*

To prove this claim, suppose that σ is a simplex whose vertices lie in L_0. Set $L = L_0 \cup |\sigma|$. It is clear that L satisfies condition (i) of the definition of a good complex, since L_0 does, and that $\Theta(L) = \Theta(L_0)$. Hence L is good; by the maximality of L_0 we have $L_0 = L$, so that $\sigma \in L_0$. This proves Claim 4.3.1.

4.3.2. Claim. *If e is any 1-simplex of L_0, then* $\mathrm{link}_K(e) \subset L_0$.

To prove this claim, let u and v denote the vertices of e, and let w be any vertex in the link of e. Let x_u, x_v and x_w be generators of X_u, X_v and X_w respectively. The vertices u, v and w span a 2-simplex σ. We set $L = L_0 \cup |\sigma|$. We shall show that L is good; by the maximality of L_0 this means that $L = L_0$, so that $w \in L_0$. Since L_0 is full in K, the claim will then follow.

Condition (i) of the definition of a good complex is clear. Condition (iii) is also clear since $\Theta(L) \geq \Theta(L_0)$. To verify condition (ii), note that $\Theta(L) = \langle \Theta(L_0) \cup \{x_w\} \rangle$. We shall apply Lemma 4.2, with $A = \Theta(L_0)$ and with $x = x_u, y = x_v, z = x_w$, to show that $\Theta(L)$ has local rank 2.

By the hypothesis of the proposition, the group $\Theta(|e|) = \langle x_u, x_v \rangle$ is non-abelian; that is, x_u and x_v do not commute. Since L_0 is good, Γ is 3-free over some finitely generated subgroup J of $\Theta(L_0)$. Finally, since u, v and w span a 2-simplex σ, the hypothesis of the proposition implies that $\Theta(\sigma) = \langle x_u, x_v, x_w \rangle$ is not free of rank 3, and so x_u, x_v and x_w are not independent. It now follows from Lemma 4.2 that $\Theta(L) = \langle \Theta(L_0) \cup \{x_w\} \rangle$ has local rank at most 2, and the proof of Claim 4.3.2 is complete.

We now proceed to the proof that $L_0 = K$, which will complete the proof of the proposition. Note that since K is connected by the hypothesis of the proposition, and since L_0 is full and non-empty(!), we need only prove that for any vertex v_0 of L_0 we have $\mathrm{link}_K(v_0) \subset L_0$. Set $C = \mathrm{link}_K(v_0)$ and $D = C \cap L_0$; we must show that $D = C$. Since L_0 is connected and contains more than one vertex, we must have $D \neq \emptyset$. Note also that D is a full subcomplex of C since L_0 is a full subcomplex of K. But C is also connected by the hypothesis of the proposition. Hence in order to prove that $D = C$ we need only prove that for any vertex $v \in D$ we have $\mathrm{link}_C(v) \subset D$. If e denotes the 1-simplex joining v_0 to v, we have $\mathrm{link}_C(v) = \mathrm{link}_K(e) \subset L_0$ by Claim 4.3.2, and hence $\mathrm{link}_C(v) \subset D$ as required. □

4.4. Proposition. *Let Θ be a normal subgroup of a finitely generated group Γ. Suppose that Θ has local rank 2, and that Γ is 3-free over some finitely generated subgroup of Θ. Suppose also that Θ contains an element x_0 with the property that for every element $\gamma \in \Gamma$ which is not a power of x_0, the element $\gamma x_0 \gamma^{-1}$ does not commute with x_0. Then Γ is a free group of rank 2.*

Proof. Let $\gamma_1, \ldots, \gamma_r$ be a finite generating set for Γ, and set $\Theta_k = \langle \Theta \cup \{\gamma_1, \ldots, \gamma_k\} \rangle$ for $k = 0, \ldots, r$. (In particular $\Theta_0 = \Theta$.)

4.4.1. Claim. *The group Θ_k has local rank 2 for $k = 0, \ldots, r$.*

By hypothesis, this claim holds for $k = 0$. We proceed by induction on k. Suppose that $0 < k \leq r$ and that Θ_{k-1} has local rank 2. Since Θ is normal in Γ and contains x_0, the elements x_0 and $\gamma_k x_0 \gamma_k^{-1}$ belong to Θ and hence to Θ_{i-1}. We now wish to apply Lemma 4.2, taking $x = x_0$, $y = \gamma_k x_0 \gamma_k^{-1}$, $z = \gamma_k$ and $A = \Theta_{k-1}$. By hypothesis Γ is 3-free over some finitely generated subgroup J of Θ. It is obvious that x, y and z are not independent. Thus if x and y do not commute, Lemma 4.2 guarantees that $\Theta_k = \langle \Theta_{k-1} \cup \{\gamma_k\}\rangle$ has local rank ≤ 2, and the induction is complete in this case.

There remains the case in which $x = x_0$ and $y = \gamma_k x_0 \gamma_k^{-1}$ commute. In this case, the property of x_0 given in the hypothesis of the theorem implies that γ_k is a power of x_0. But in this case we have $\gamma_k \in \Theta \leq \Theta_{k-1}$, so that $\Theta_k = \Theta_{k-1}$, and the induction step is trivial. The proof of Claim 4.4.1 is therefore complete.

It is clear from the definition of the Θ_k that $\Theta_r = \Gamma$. Applying Claim 4.4.1 with $k = r$ we conclude that Γ has local rank at most 2. Since Γ is finitely generated this means $\operatorname{rank}\Gamma \leq 2 < 3$. But by hypothesis Γ is 3-free over some finitely generated subgroup of Θ. Hence Γ is a free group. Since its local rank is 2, it is in fact free of rank 2. $\square$

Section 5. The proof of Theorem B

5.1. The goal of this section is to prove the following theorem.

Theorem. *Let M be a closed hyperbolic manifold of dimension $n \geq 3$. Let us write $M = \mathbf{H}^n/\Gamma$, where Γ is a co-compact, discrete, torsion-free group of isometries of $\mathbf{H}^n$. Set*

$$\lambda = \frac{\log 5}{n-1},$$

and suppose that the following condition holds:

- ($\star$) *If x and y are non-commuting elements of Γ such that $Z_\lambda(x) \cap Z_\lambda(y) \neq \emptyset$, then Γ is 3-free over $\langle x, y\rangle$.*

Then M contains a hyperbolic ball of radius $\lambda/2$.

5.2. As we observed in 4.1, a group which is k-free is k-free over any subgroup. Hence condition ($\star$) of Theorem 5.1 always holds if $\Gamma \cong \pi_1(M)$ is 3-free. Thus Theorem B of the Introduction is a special case of Theorem 5.1. In Section 6 we will show how to deduce Theorem A of the Introduction from Theorem 5.1.

Proof of Theorem 5.1. Suppose that M satisfies the hypotheses of Theorem 5.1 but contains no ball of radius $\lambda/2$, where

$$\lambda = \frac{\log 5}{n-1}.$$

Then in the notation of 3.4 we have a covering $(Z_\lambda(X))_{X \in \mathcal{X}}$ of $\mathbf{H}^n$ with index set $\mathcal{X} = \mathcal{X}_\lambda(M)$ and nerve $K = K_\lambda(M)$. By definition the vertices of K are in natural one-one correspondence with the maximal cyclic subgroups in the set $\mathcal{X}$. If we denote by $X_v \in \mathcal{X}$ the maximal cyclic subgroup corresponding to a vertex v, then $(K, (X_v)_v)$ is a Γ- labeled complex in the sense of 4.3.

We shall show that $(K, (X_v)_v)$ satisfies the hypotheses of Proposition 4.3. By Proposition 3.4, K is a connected simplicial complex with more than one vertex, and the link of every vertex of K is connected. Now let e be any 1-simplex of K, and let v and w denote its vertices. Let x_v and x_w be generators of X_v and X_w. We have $v \neq w$ and hence $X_v \neq X_w$; hence by 3.1 the elements x_v and x_w do not commute, and the group $\Theta(|e|) = \langle x_v, x_w \rangle$ is non-abelian. On the other hand, by the definition of the nerve K we have $Z_\lambda(X_v) \cap Z_\lambda(X_w) \neq \emptyset$, and so the hypothesis of the Theorem implies that Γ is 3-free over $\langle x_v, x_w \rangle$. Finally, let σ be any 2-simplex of K, and let u, v and w denote its vertices. Let x_u, x_v and x_w be generators of X_u, X_v and X_w. By the definition of the nerve K we have $Z_\lambda(X_u) \cap Z_\lambda(X_v) \cap Z_\lambda(X_w) \neq \emptyset$. Hence by Proposition 3.5, the elements x_u, x_v and x_w are not independent. By [**6**, p. 59] this means that $\Theta(|\sigma|) = \langle x_u, x_v, x_w \rangle$ is not a free group of rank 3.

Thus Proposition 4.3 applies and we conclude that $\Theta(K)$ has local rank 2. We claim that $\Theta = \Theta(K)$ in fact satisfies all the hypotheses of Proposition 4.4. To show that Θ is normal, observe that by definition Θ is generated by all the maximal subgroups in $\mathcal{X}$. If a maximal cyclic subgroup X belongs to $\mathcal{X}$, i.e. if $Z_\lambda(X) \neq \emptyset$, then for any $\gamma \in \Gamma$ we have $Z_\lambda(\gamma X \gamma^{-1}) = \gamma \cdot Z_\lambda(X) \neq \emptyset$. Thus Θ is a normal subgroup of Γ.

We saw above that for any edge $|e|$ of K the group Γ is 3-free over the 2-generator subgroup $\Theta(|e|)$ of $\Theta(K)$. The only hypothesis of Proposition 4.4 left to check is the existence of the element x_0. We take x_0 to be a generator of any group $X_0 \in \mathcal{X}$. If γ is an element of Γ such that $x = x_0$ and $y = \gamma_k x_0 \gamma_k^{-1}$ commute, then by 3.1, the elements x_0 and $\gamma_k x_0 \gamma_k^{-1}$ generate the same maximal cyclic subgroup of Γ, so $\gamma_k x_0 \gamma_k^{-1} = x_0^{\pm 1}$. Hence γ_k^2 commutes with x_0. Thus γ_k and x_0 belong to $C(\gamma_k^2)$, which by 3.1 is a maximal cyclic subgroup containing x_0 and is therefore generated by x_0. Hence γ_k is a power of x_0.

It now follows from Lemma 4.4 that Γ is a free group of rank 2. However, this is impossible, because Γ, as the fundamental group of a closed hyperbolic n-manifold, must have cohomological dimension $n \geq 3$, whereas a free group has cohomological dimension 1. This contradiction completes the proof of Theorem 5.1. $\square$

Section 6. The proof of Theorem A

6.1. We shall prove the following result.

Theorem. *Let M be a closed orientable hyperbolic 3-manifold. Suppose that every subgroup of $\pi_1(M)$ whose rank is at most 3 is of infinite index in $\pi_1(M)$. Then M contains a hyperbolic ball of radius* $(\log 5)/4$.

6.2. Now recall the statement of [**8**, Proposition 1.1]. Let M be a closed 3-manifold, let p be a prime number, and let k be a positive integer. Suppose that either M is orientable or $p = 2$. Suppose that the $\mathbf{Z}_p$-vector space $H_1(M; \mathbf{Z}_p)$ has dimension at least $k+2$. Then every subgroup of $\pi_1(M)$ having rank $\leq k$ is of infinite index.

In particular, if M is a closed, orientable, hyperbolic 3-manifold, and if $H_1(M; \mathbf{Z}_p)$ has dimension at least 5 for some prime p, then every subgroup of $\pi_1(M)$ having rank ≤ 3 is of infinite index. Combining this with Theorem 6.1 we obtain Theorem A of the Introduction.

6.3. It remains to give the

Proof of Theorem 6.1. We can write $M = \mathbf{H}^3/\Gamma$, where Γ is a co-compact, discrete, torsion-free group of orientation-preserving isometries of $\mathbf{H}^3$. Recall that since Γ is co-compact, every non-trivial element of Γ is loxodromic. In particular Γ contains no parabolic elements. Set

$$\lambda = \frac{\log 5}{2}.$$

We wish to apply Theorem 5.1 to conclude that M contains a hyperbolic ball of radius $\lambda/2$. It suffices to show that condition $(\star)$ of 5.1 holds.

Suppose that x and y are non-commuting elements of Γ such that $Z_\lambda(x) \cap Z_\lambda(y) \neq \emptyset$. We must show that Γ is 3-free over $\langle x, y\rangle$. Let Θ be a subgroup of rank ≤ 3 containing x and y. We are required to prove that Θ is free.

The hypothesis of the theorem guarantees that Θ has infinite index in $\Gamma \cong \pi_1(M)$. Thus Θ is not co-compact.

Let us choose a point $P_0 \in Z_\lambda(x) \cap Z_\lambda(y)$. By the definition of $Z_\lambda(x)$ and $Z_\lambda(y)$ there exist positive integers a and b such that $D_{x^a}(P_0) < \lambda$ and $D_{y^b}(P_0) < \lambda$; that is,

$$\max(\text{dist}(P_0, x^a(P_0)), \text{dist}(P_0, y^b(P_0))) < \lambda.$$

We observe that x^a and y^b do not commute. Indeed, it follows from 3.1 that $C(x^a) = C(x)$ and that $C(y^b) = C(y)$. Hence if x^a and y^b were to commute then x and y would also commute, which they do not.

Before showing that Θ is free we will show that it is *freely decomposable,* i.e. that it is a free product of two non- trivial subgroups. Assume to the contrary that Θ is freely indecomposable. According to [**1**], if Θ is any freely indecomposable, discrete, torsion-free group of orientation- preserving isometries of $\mathbf{H}^3$, then Θ is *topologically tame*, that is, the quotient hyperbolic 3-manifold $\mathbf{H}^3/\Theta$ is homeomorphic to the interior of a compact 3-manifold with boundary. Furthermore, according to [**2**, Proposition 3.2], if Θ is any non-co-compact, topologically tame, discrete, torsion-free group of orientation-preserving isometries of $\mathbf{H}^3$, then any finitely generated subgroup of Θ is topologically tame.

In particular, $\langle x^a, y^b\rangle$ is topologically tame. Of course, since Θ is non-co-compact, $\langle x^a, y^b\rangle$ is also non-co-compact. We now recall the statement of the main theorem of [**3**]. Let ξ and η be non-commuting orientation-preserving isometries of $\mathbf{H}^3$. Suppose that $\langle \xi, \eta\rangle$ is discrete, torsion-free, topologically tame and non-co-compact, and contains no parabolic elements. Then for any point $P \in \mathbf{H}^3$ we have

$$\max(\text{dist}(P, \xi(P)), \text{dist}(P, \eta(P))) \geq \log 3.$$

As we have checked the hypotheses of this statement for $\xi = x^a$ and $\eta = y^b$ we now have

$$\max(\text{dist}(P_0, x^a(P_0)), \text{dist}(P_0, y^b(P_0))) \geq \log 3.$$

Since

$$\lambda = \frac{\log 5}{2} < \log 3,$$

we have a contradiction. This proves that Θ is freely decomposable.

Thus we may write $\Theta = \Theta_1 * \Theta_2$, where the Θ_i are non-trivial. By Grushko's theorem [**9**] we have $\operatorname{rank}\Theta_1 + \operatorname{rank}\Theta_2 = \operatorname{rank}\Theta \leq 3$, and hence each Θ_i has rank at most 2. But each Θ_i has infinite index in $\Gamma = \pi_1(M)$, since Θ does; and it follows from [**5**, Theorem VI.4.1] that any infinite-index subgroup of rank ≤ 2 in the fundamental group of the closed, orientable hyperbolic 3-manifold M is free. It follows that Θ_1 and Θ_2 are free, and hence that Θ is free also. This completes the proof. □

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