Planar Graphs

Marc Culler

graphs

A (finite) *graph* G is a topological space with $G = V \dot{\cup} E$ where

- V is a finite discrete set (*vertices*);
- E is a finite disjoint union of open sets (*edges*);
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Lemma. *A graph which is not a cycle is homeomorphic to a graph without valence* 2 *vertices.*

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Theorem (Jordan-Schönflies)**.** *A simple closed curve in S^2 is the common boundary of two disks with disjoint interiors.*

Theorem. *Suppose that f is a conformal homeomorphism from the open unit disk onto an open set $\Omega \subset S^2$. If the boundary of Ω is locally connected, then f extends to a continuous map defined on the closed unit disk.*

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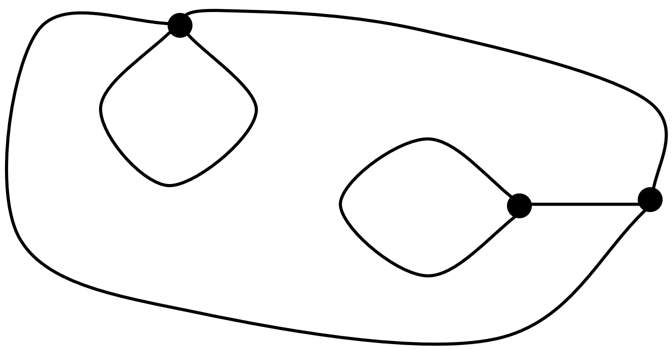

But the boundary of a face is not necessarily a cycle.

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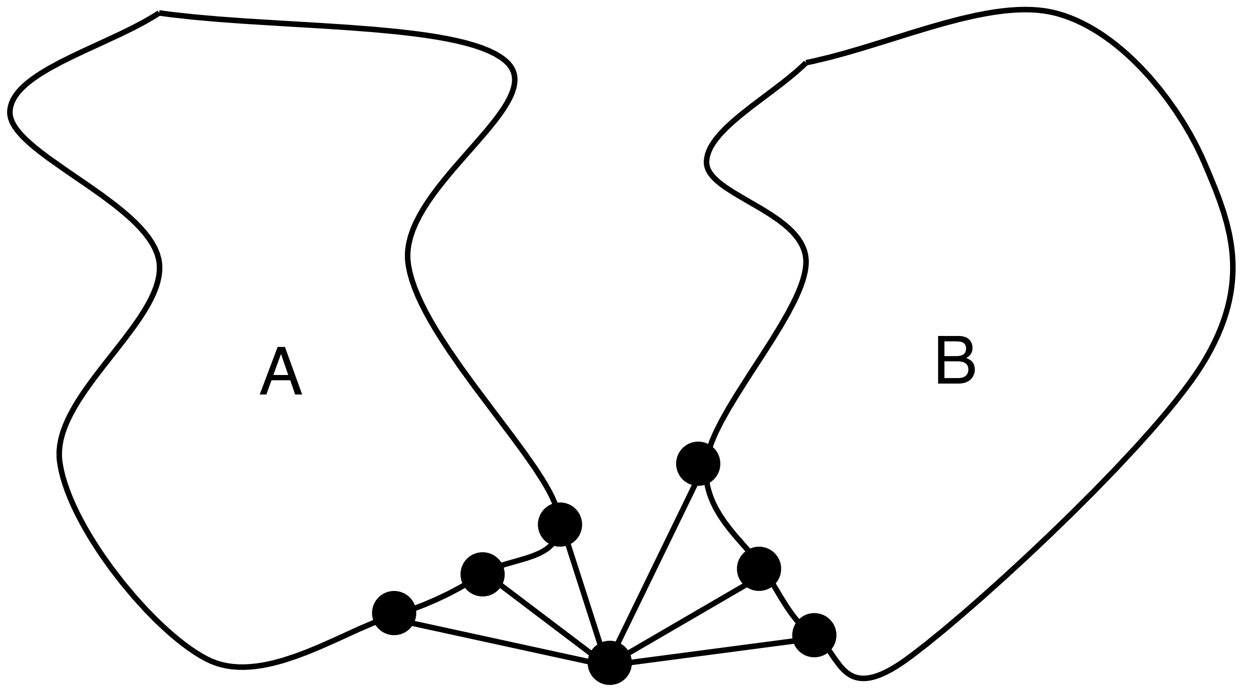

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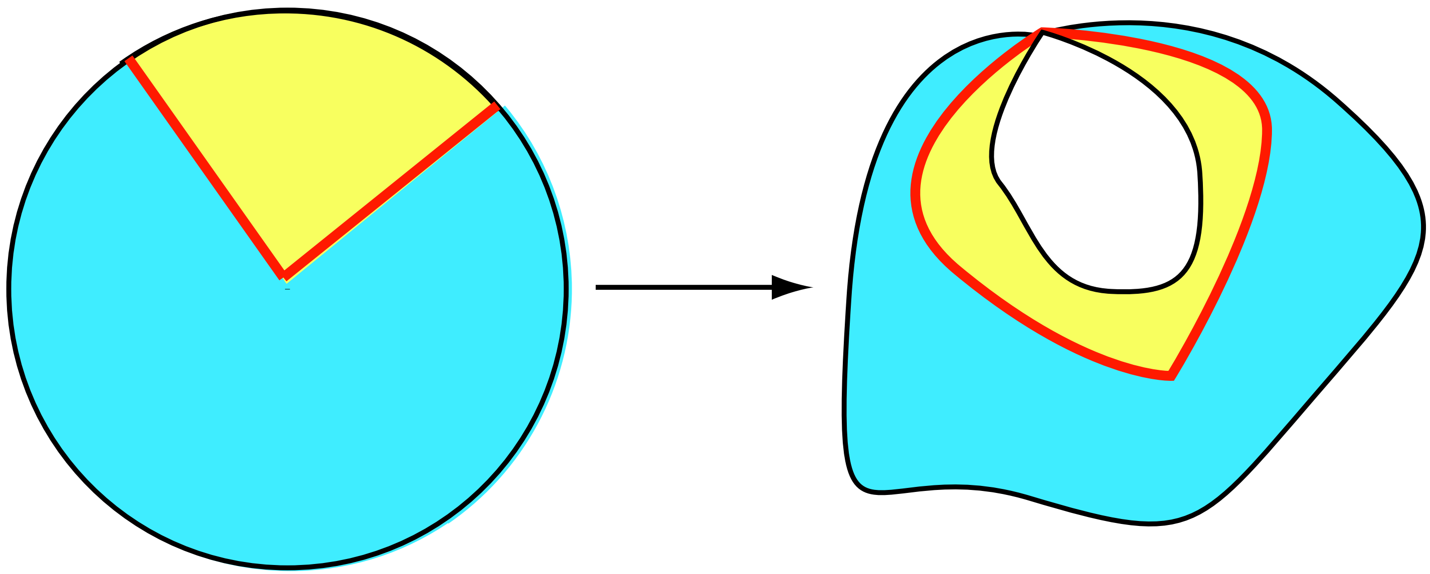

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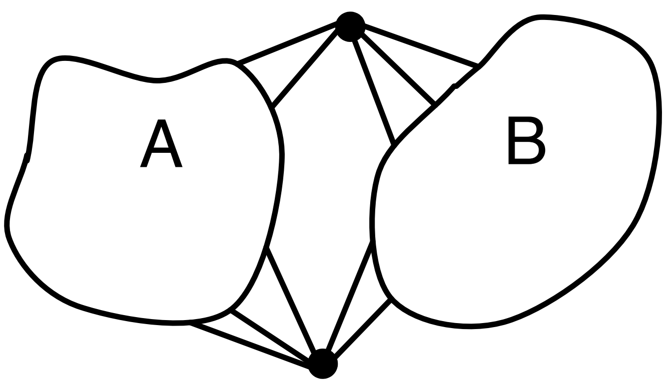

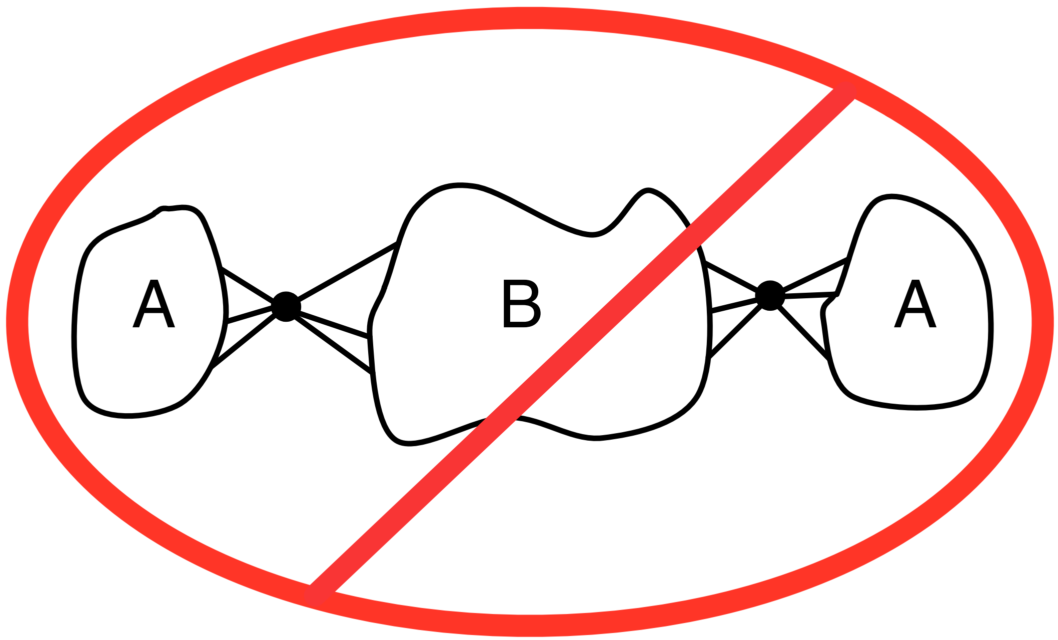

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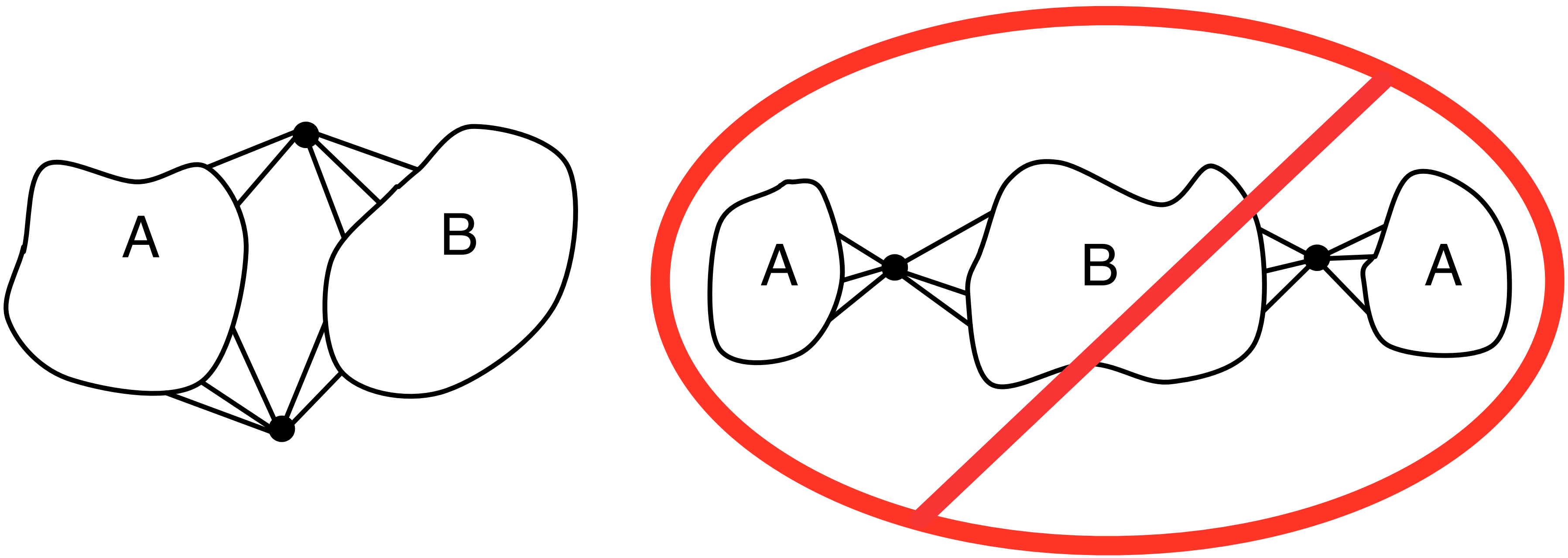

A graph is 3-*connected* if it is connected, has no cut vertex and has no cut pair.

boundaries of faces

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Proof. If $G - C$ is connected, then for any embedding of G in S^2, the connected set $G - C$ is contained in one of the two disks bounded by C. The other disk must be a face.

Suppose $G - C$ is disconnected. Write G as $A \cup B$ where A and B are subgraphs, neither one a cycle, such that $A \cap B = C$. Choose an embedding of G in S^2. If C is not the boundary of a face, then we are done. Otherwise, restrict the embeddings to A and B, to obtain embeddings of A and B into disks, sending C to the boundary of each disk. Gluing the boundaries of the two disks together gives an embedding of G in S^2 for which C is not a face. □

unique embeddings

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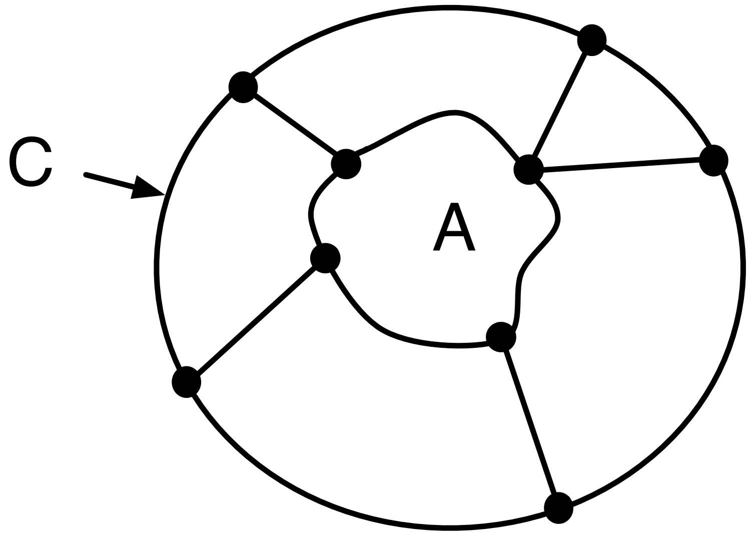

A component of $G - C$ is in the complement of the face bounded by C.

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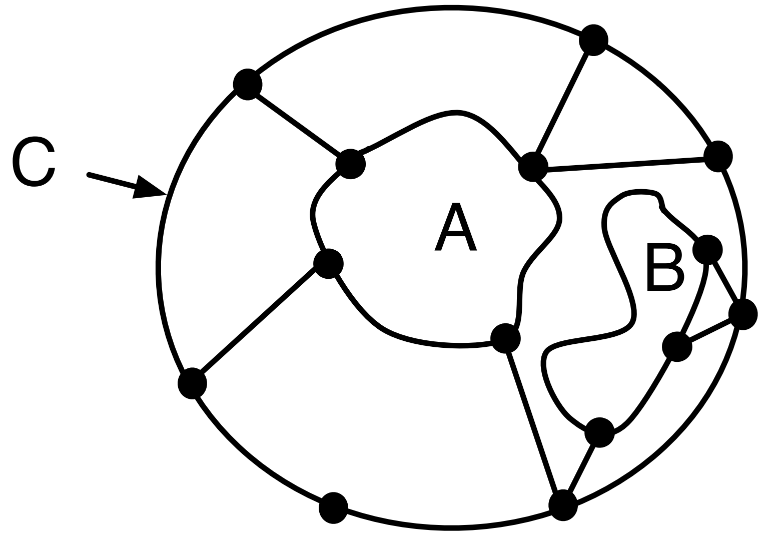

The other components of $G - C$ have to fit in the "gaps".

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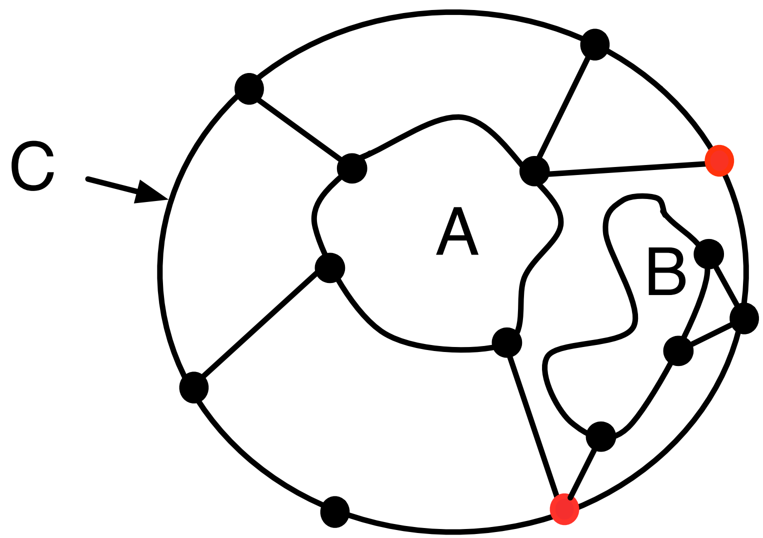

Here is a cut pair. □

Kuratowski's Theorem

A *minimal non-planar graph* is not planar, but every proper subgraph is planar.

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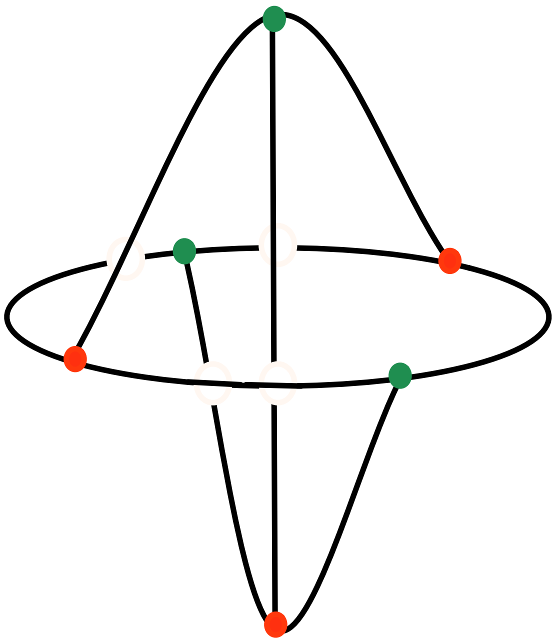

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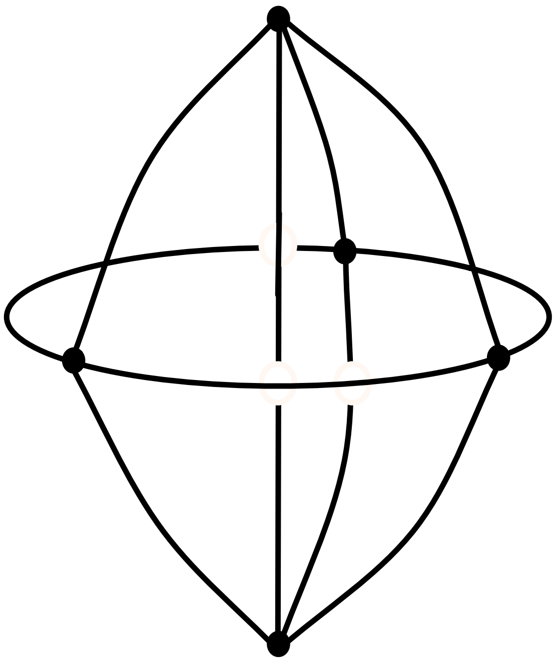

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If every face has at least k edges on its boundary then $kF \leq 2E$, so

$$2 = V - E + F \leq V - E + \frac{2}{k}E \Rightarrow E \leq \frac{k}{k-2}V - \frac{2k}{k-2}$$

If $k = 3$ then $E \leq 3V - 6$. If $k = 4$ then $E \leq 2V - 4$.

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If $k = 3$ then $E \leq 3V - 6$. If $k = 4$ then $E \leq 2V - 4$.

For $K(5)$ we can take $k = 3$ and we have $V = 5$ but $E = 10 > 15 - 6$.

For $K(3,3)$ we can take $k = 4$ and we have $V = 6$ but $E = 9 > 12 - 4$.

So these are non-planar graphs.

Lemma. *A minimal non-planar graph G has no cut vertex.*

Proof. Suppose $G = A \cup B$, $A \cap B = \{v\}$. By minimality, A and B are planar. Embed A in a closed disk, so that v lies on the boundary. Do the same for B. Then embed the two disks so they meet at v. □

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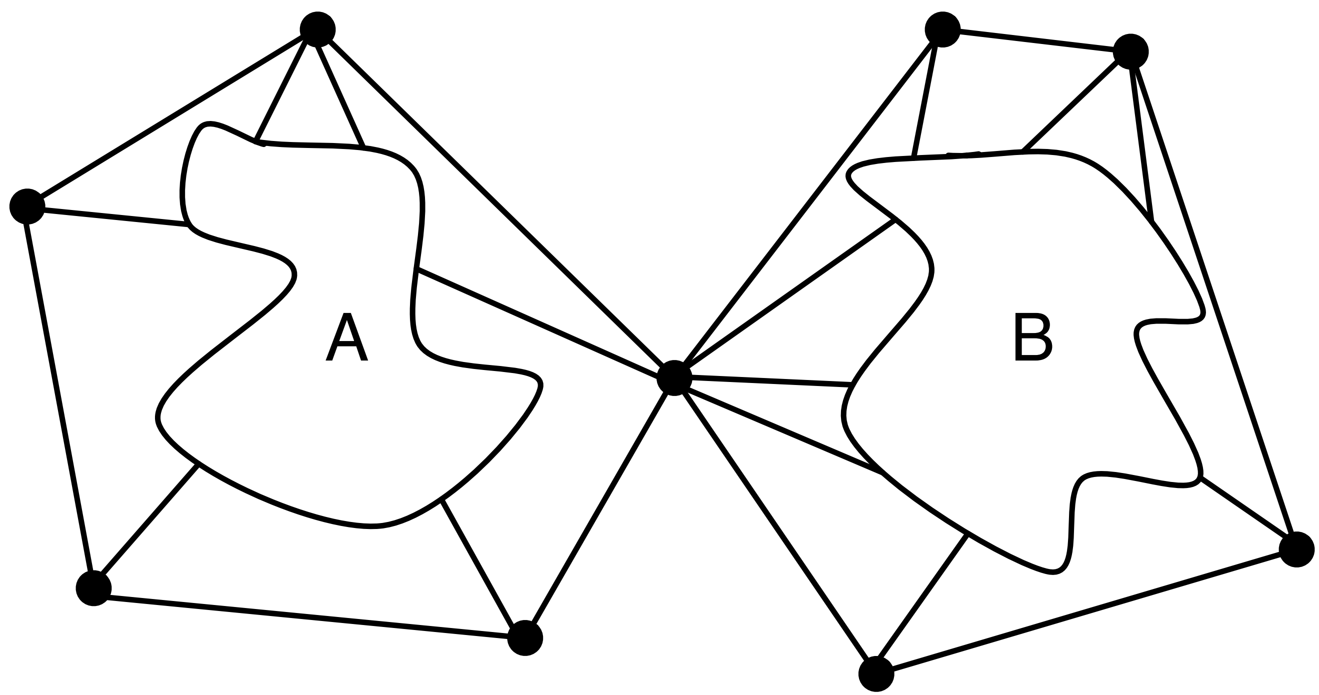

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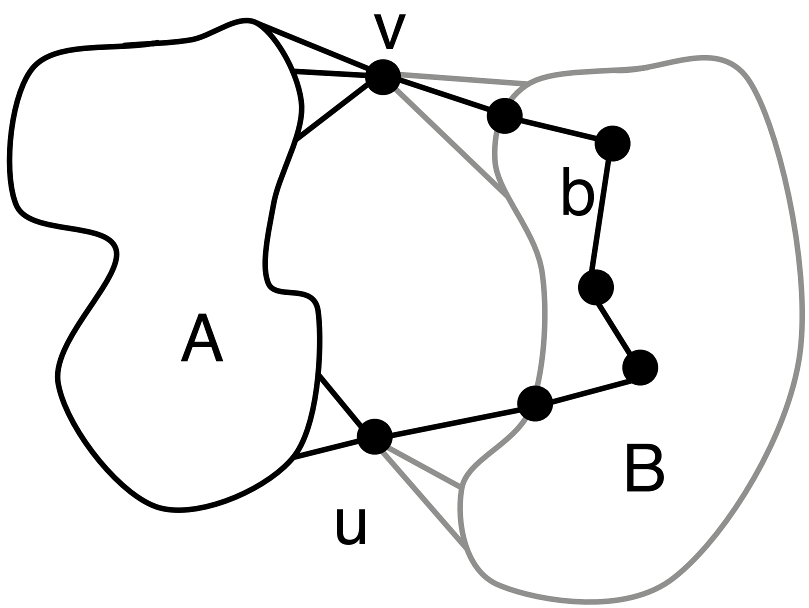

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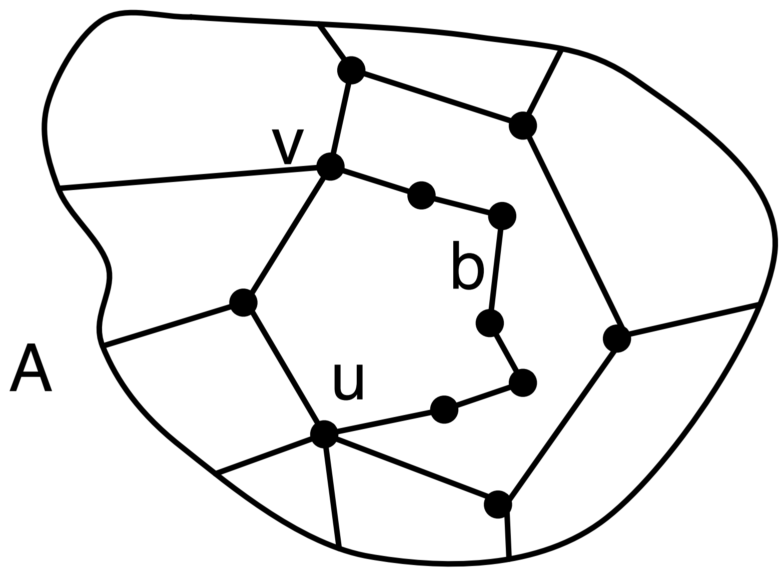

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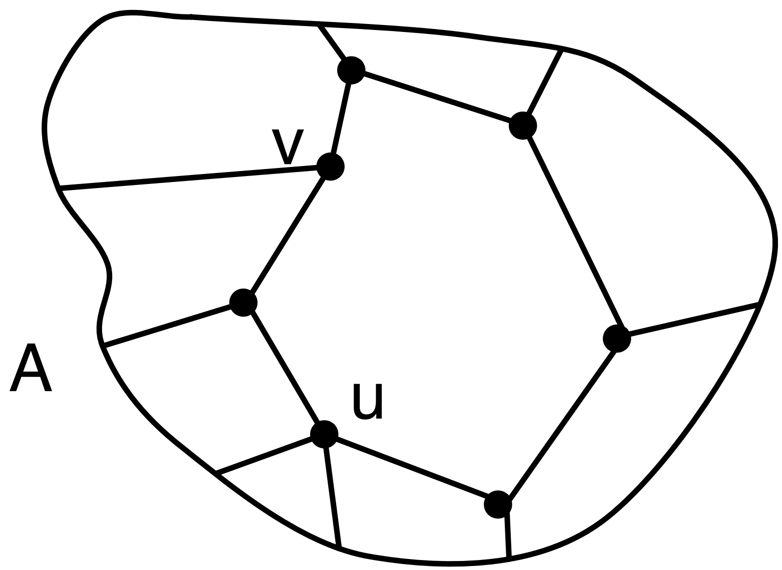
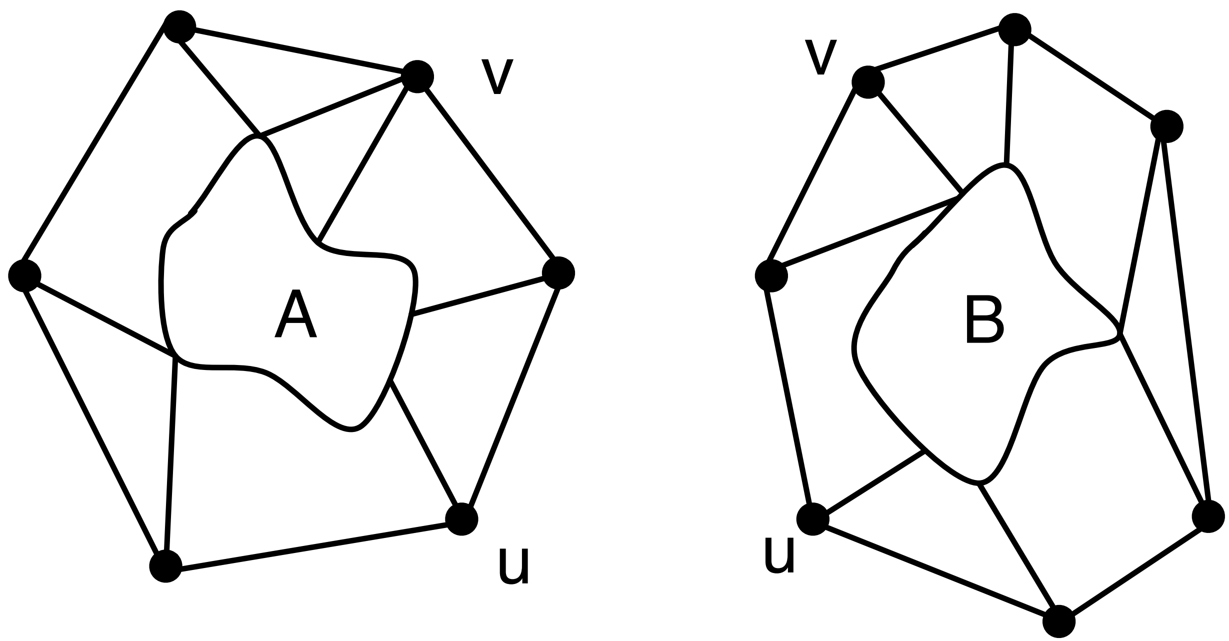


no cut pair, cont'd

To finish the proof of the lemma, embed A in a disk so that u and v lie on the boundary. Do the same for B.



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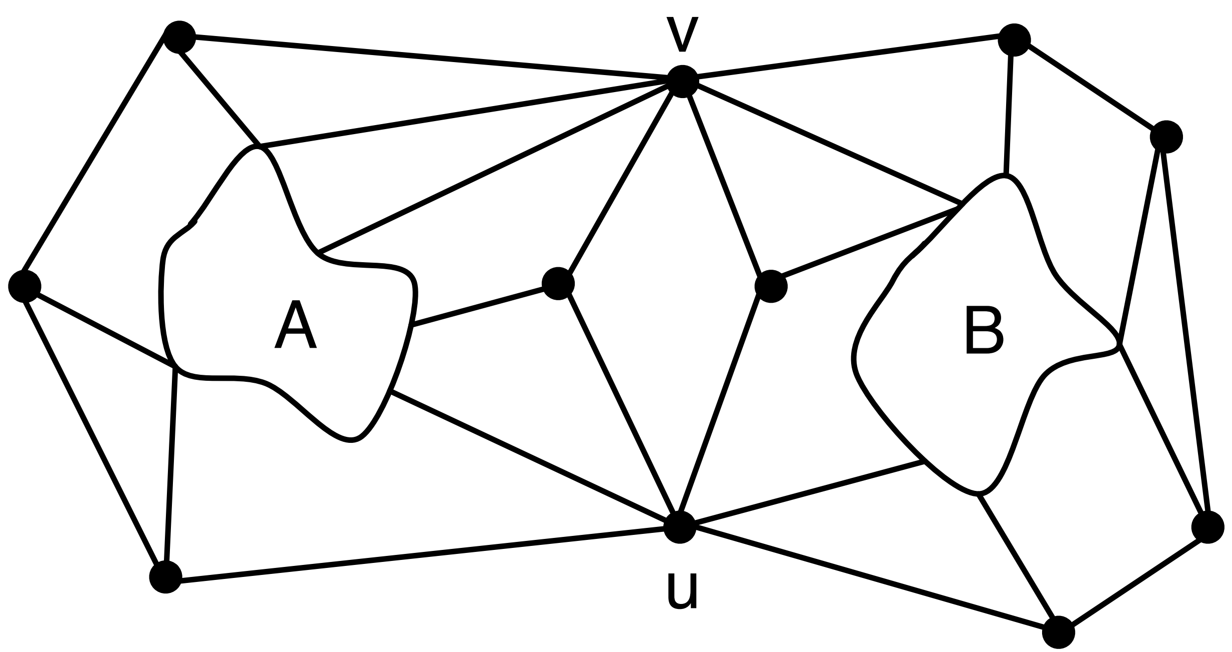

Then embed the two disks so they meet at u and v. This is a contradiction since G is non-planar.

□

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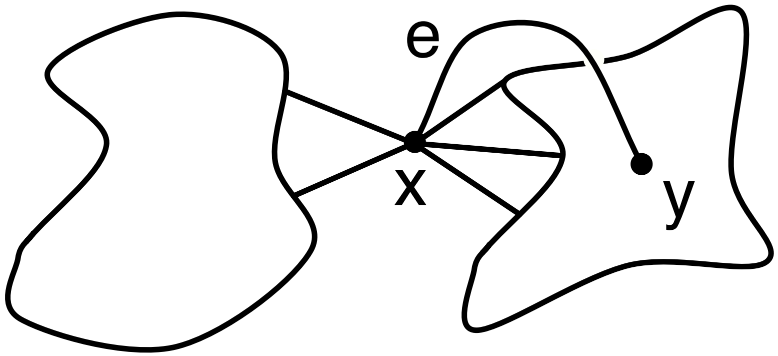

If x is a cut vertex for G', then x is a cut vertex for G. Likewise for y.

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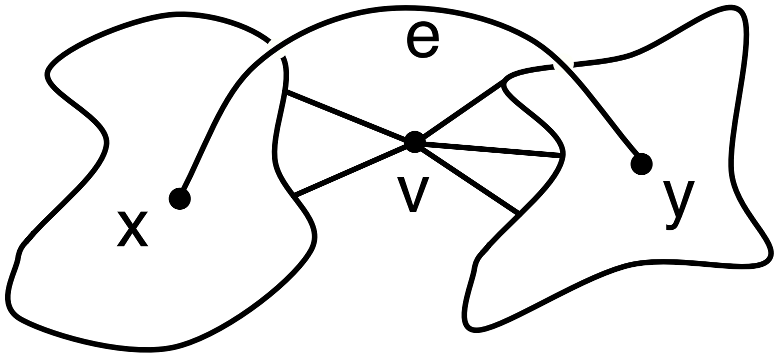

If G has a cut vertex v distinct from x and y, then x and y are separated by v and $\{x, v\}$ is a cut pair for G.

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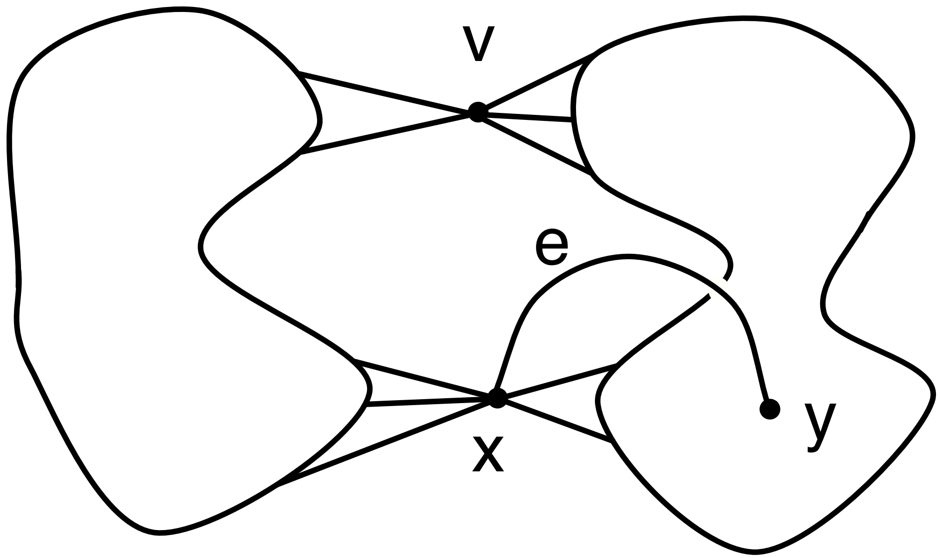
The graph G' may have cut pairs, but no cut pair can contain x.



If $\{x, v\}$ is a cut pair for G' then it is a cut pair for G as well.

the wheel

Consider the graph $G' \subset S^2$. Construct a graph $G'' \subset S^2$ by erasing the vertex x and the edges that meet it. Let R be the boundary of the face of G'' containing the point x. *Claim:* R is a cycle.

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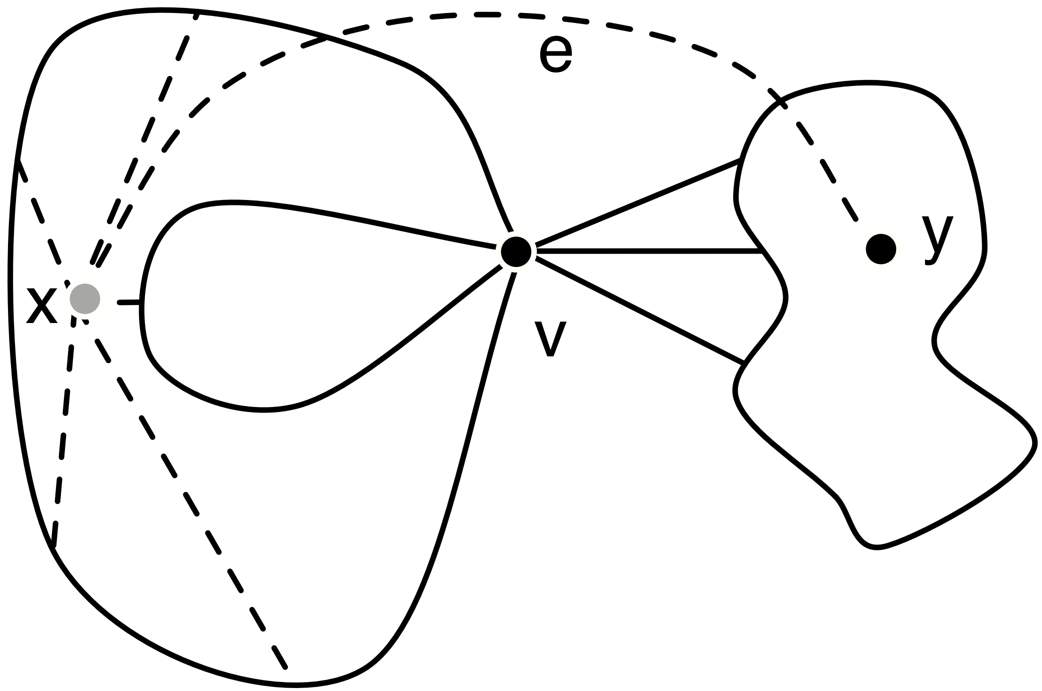

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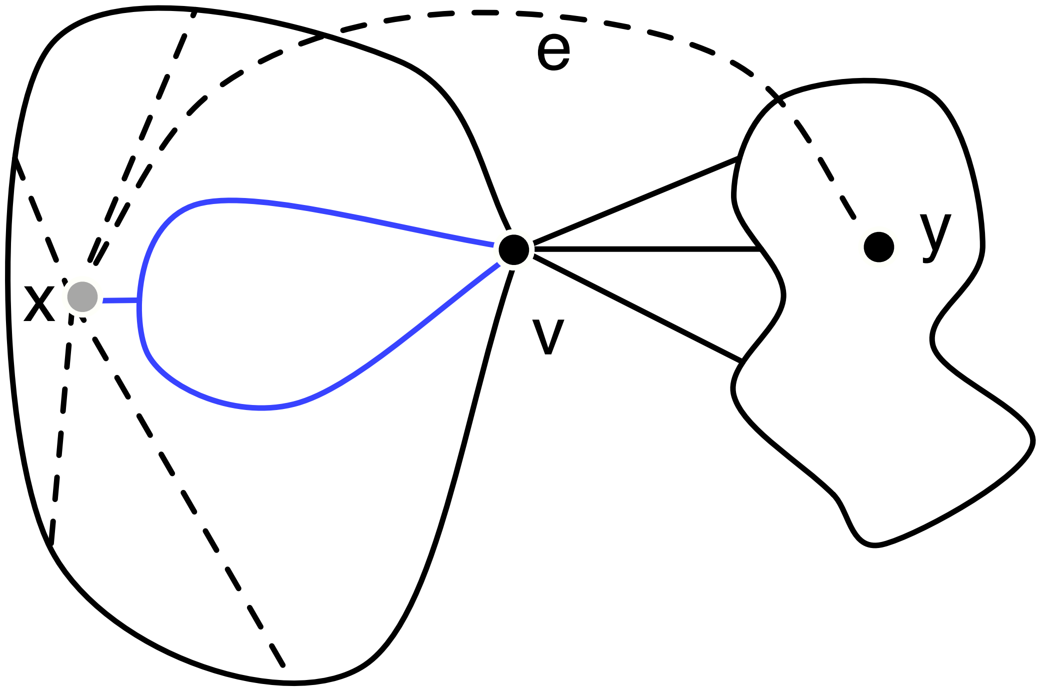

But then x and v would form a cut pair for G, a contradiction.

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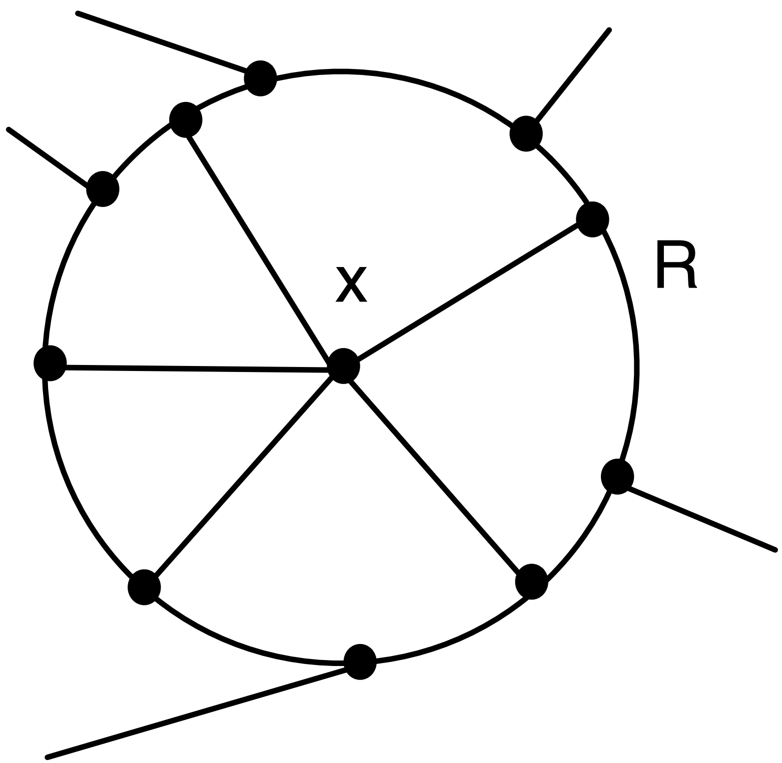

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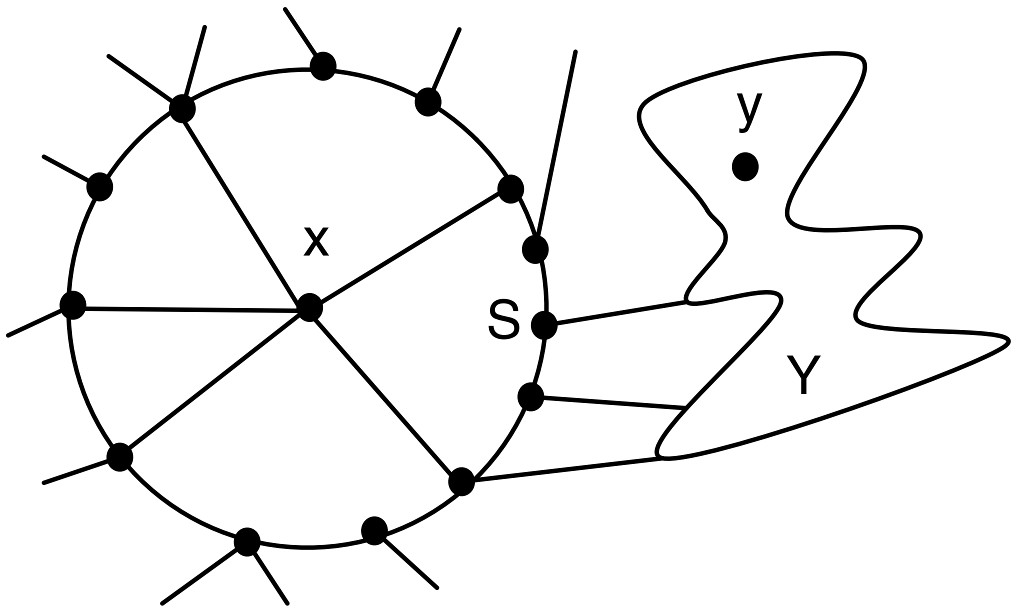

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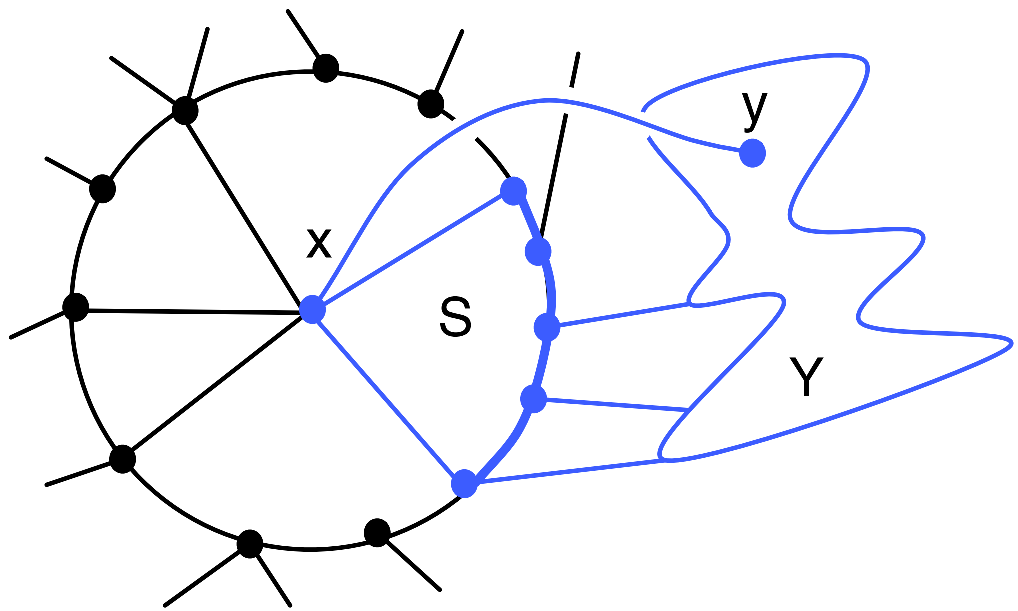

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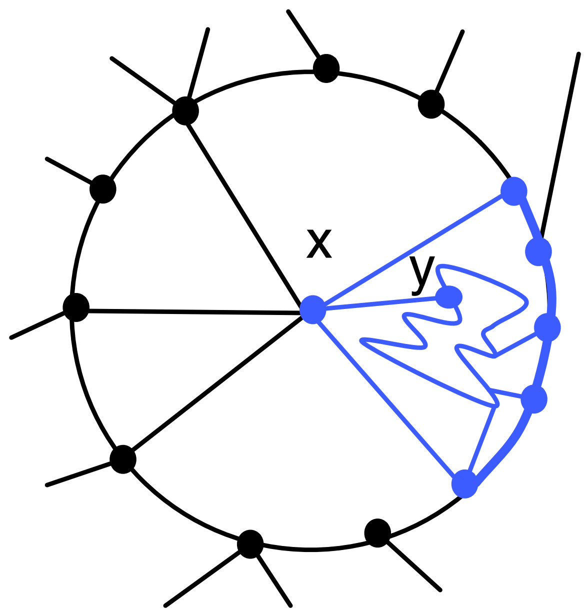

Since $Z - S$ is connected, every embedding of Z has S as a face. So we can embed Z *inside* of S. This gives a planar embedding of G, a contradiction.

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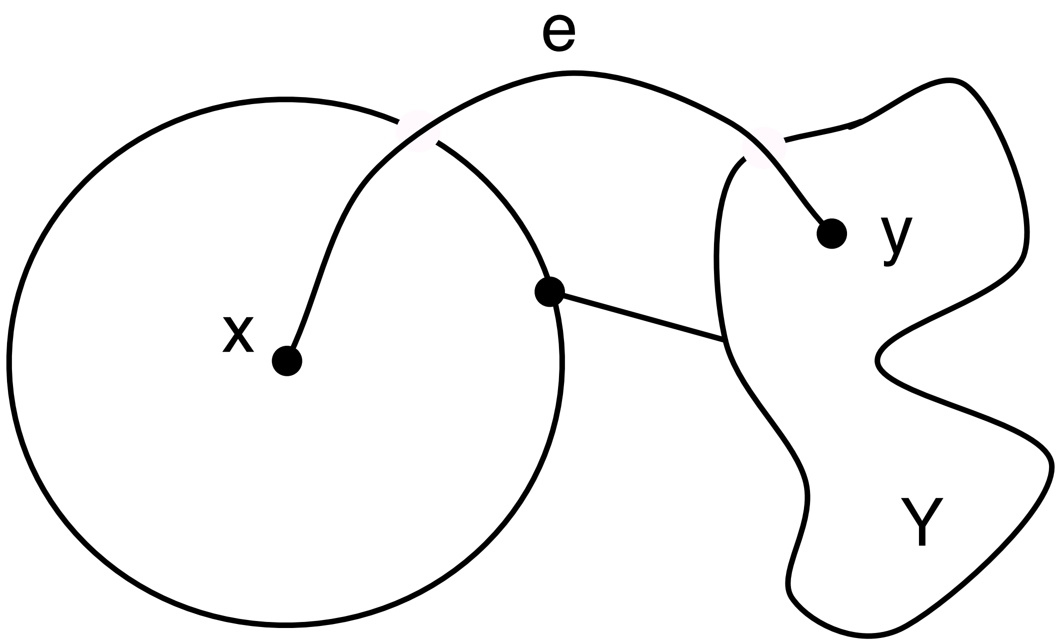

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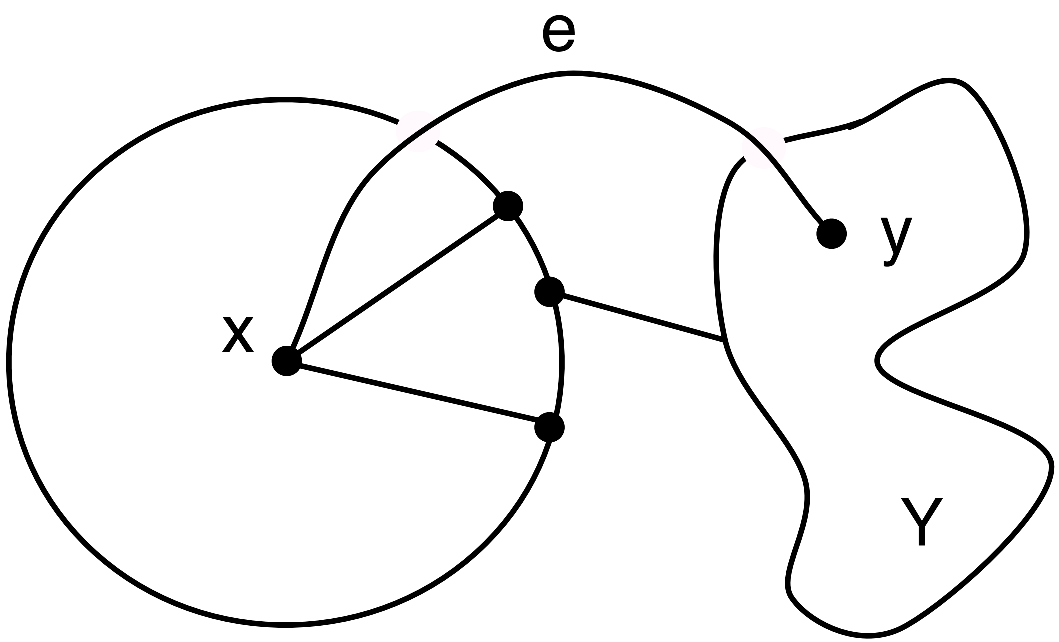

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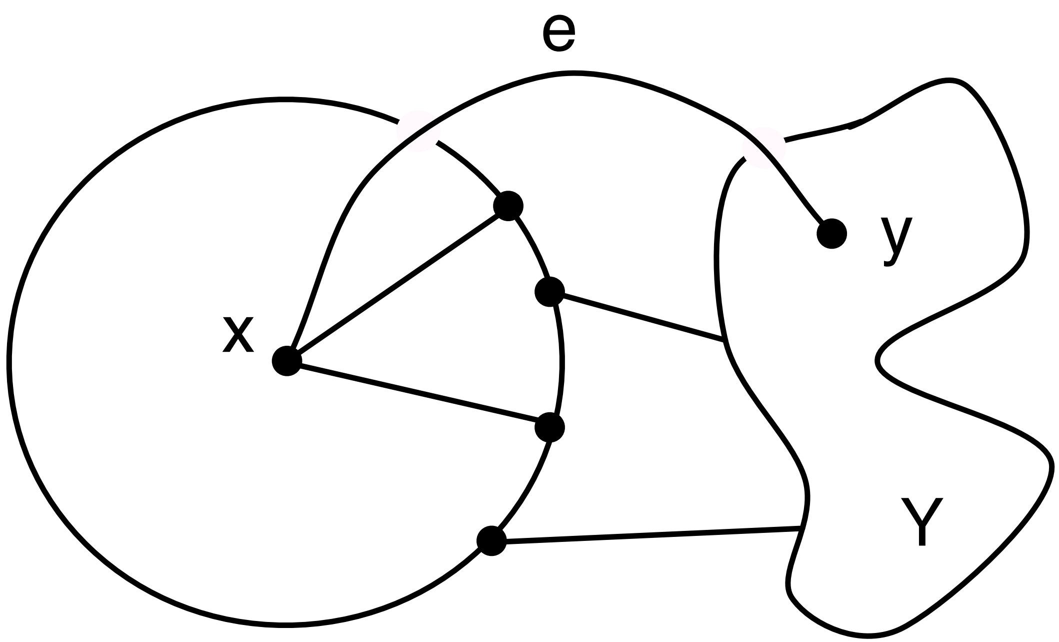

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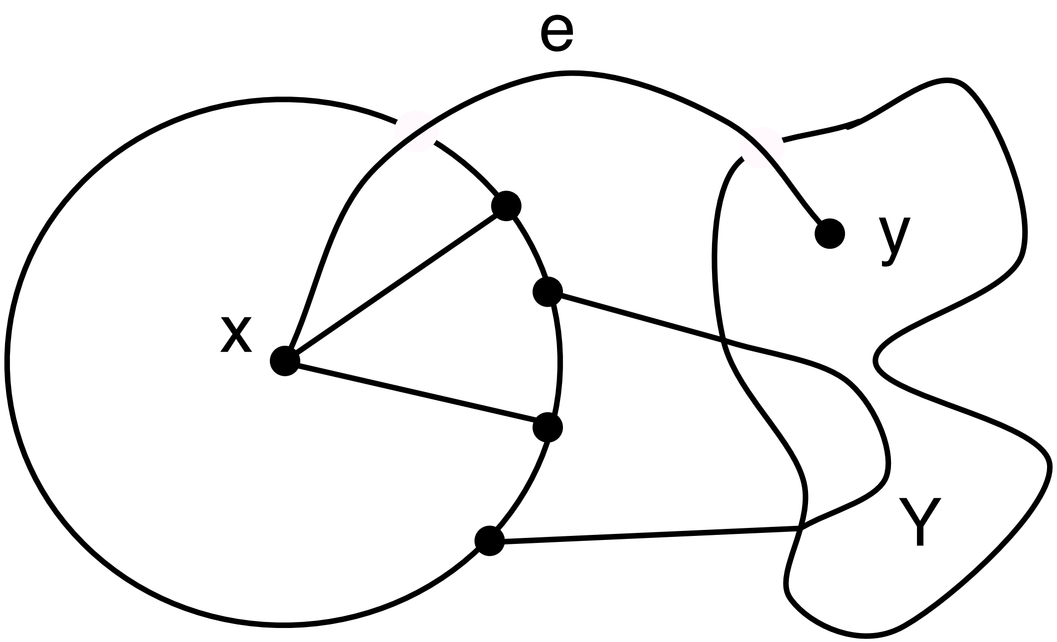

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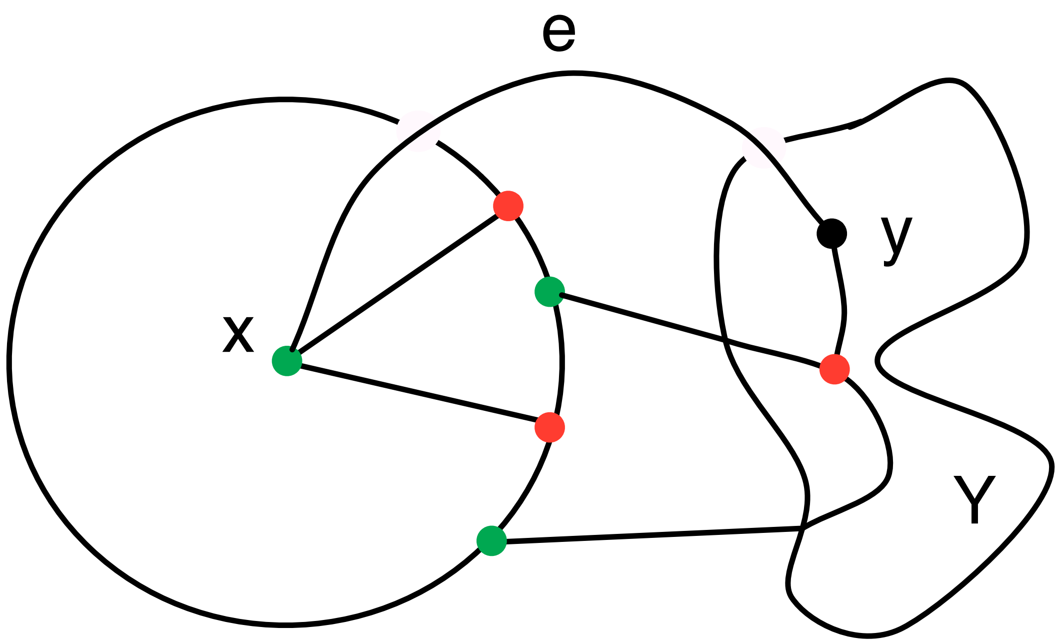

case 2

Suppose $\overline{Y}$ meets R in exactly two endpoints of spokes.

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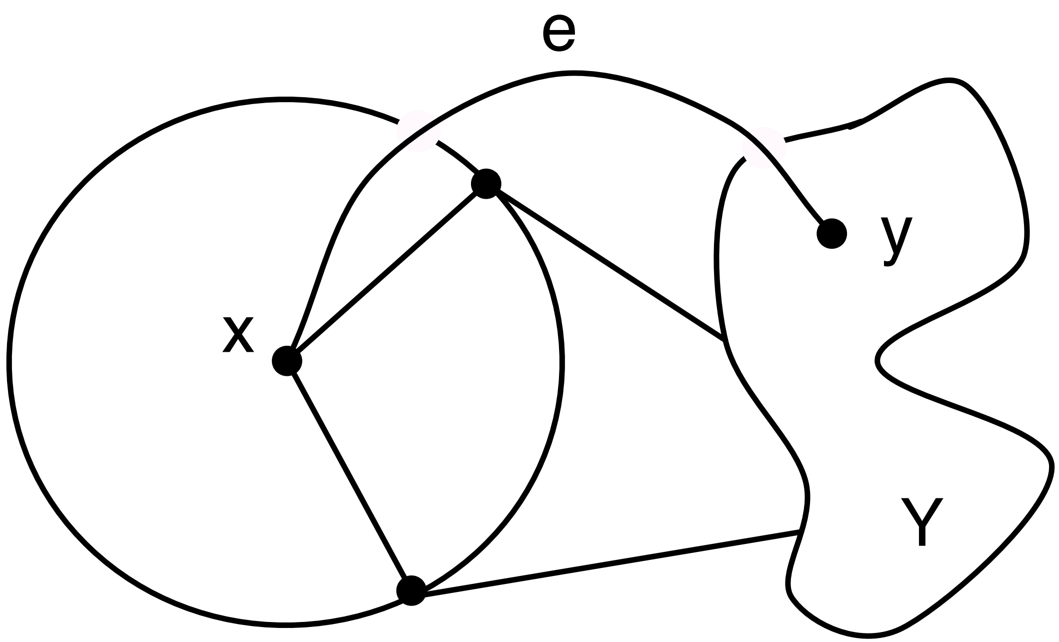

case 2

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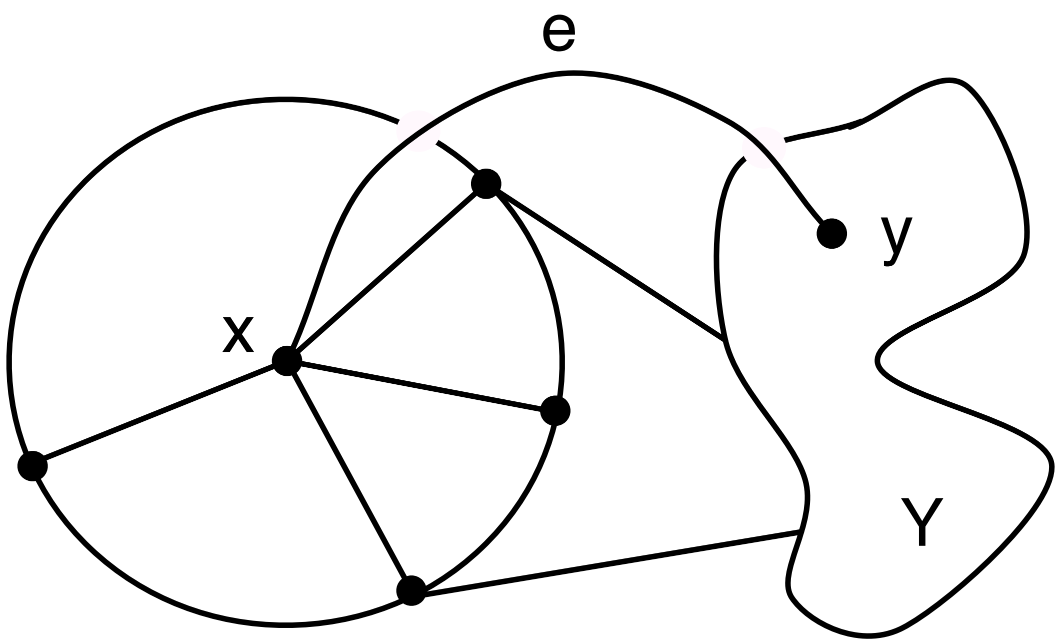
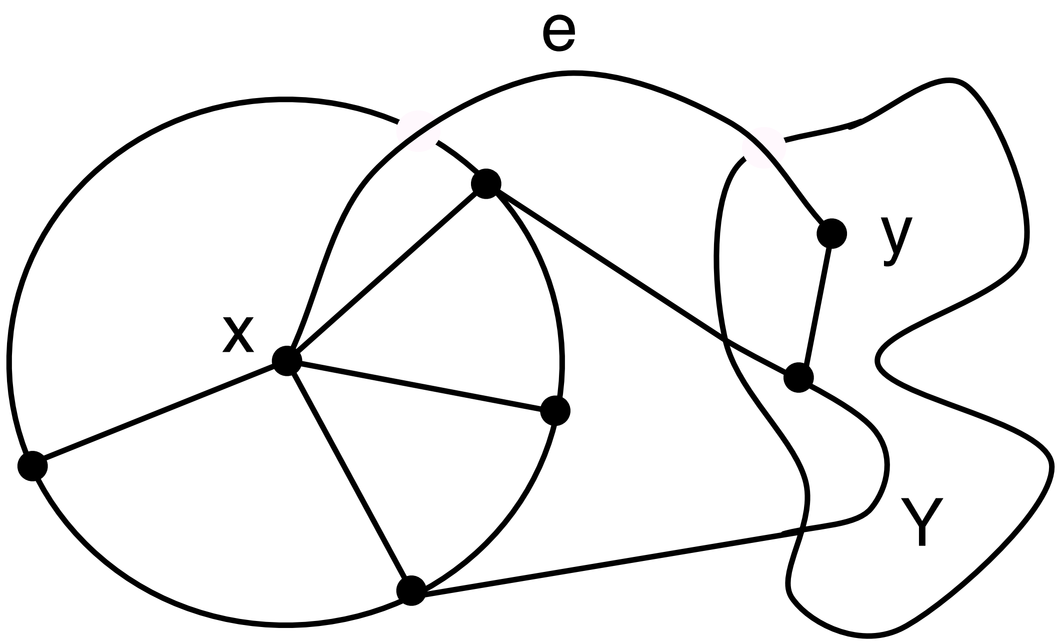


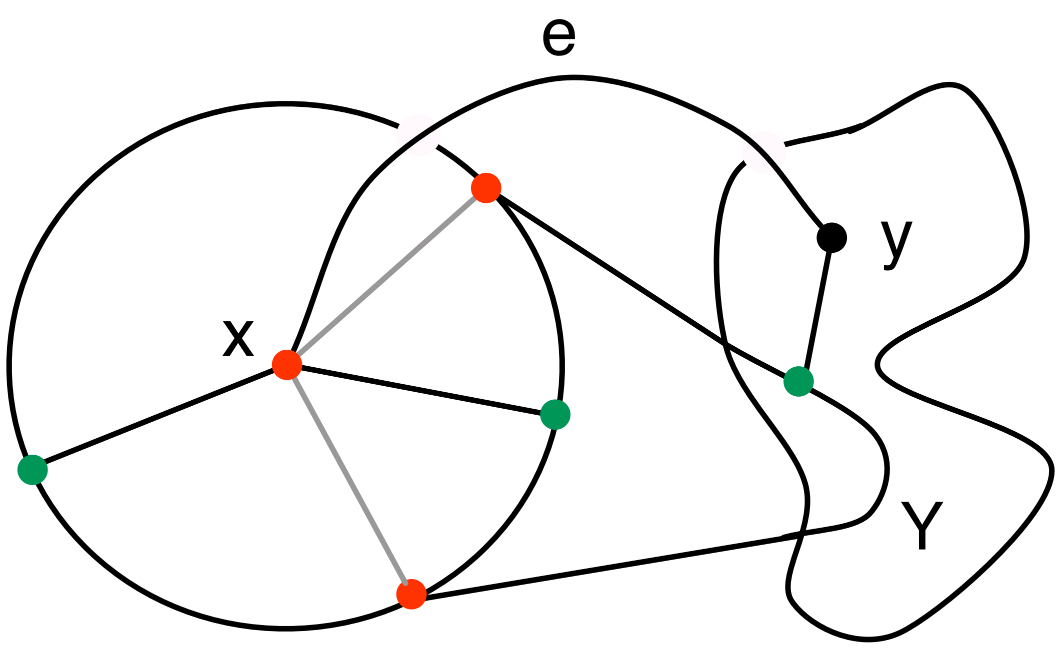
case 2

Suppose $\overline{Y}$ meets R in exactly two endpoints of spokes.



case 2

Suppose $\overline{Y}$ meets R in exactly two endpoints of spokes.



case 3

Suppose $\overline{Y}$ meets R in three or more endpoints of spokes.

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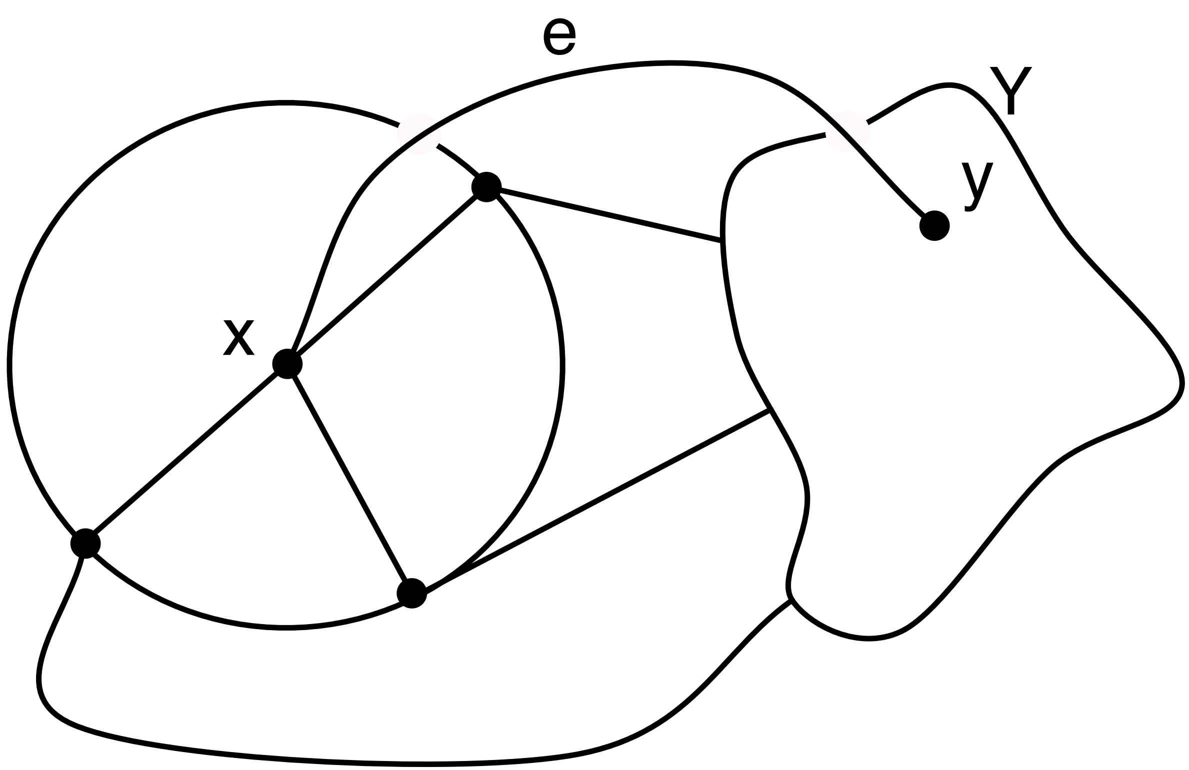
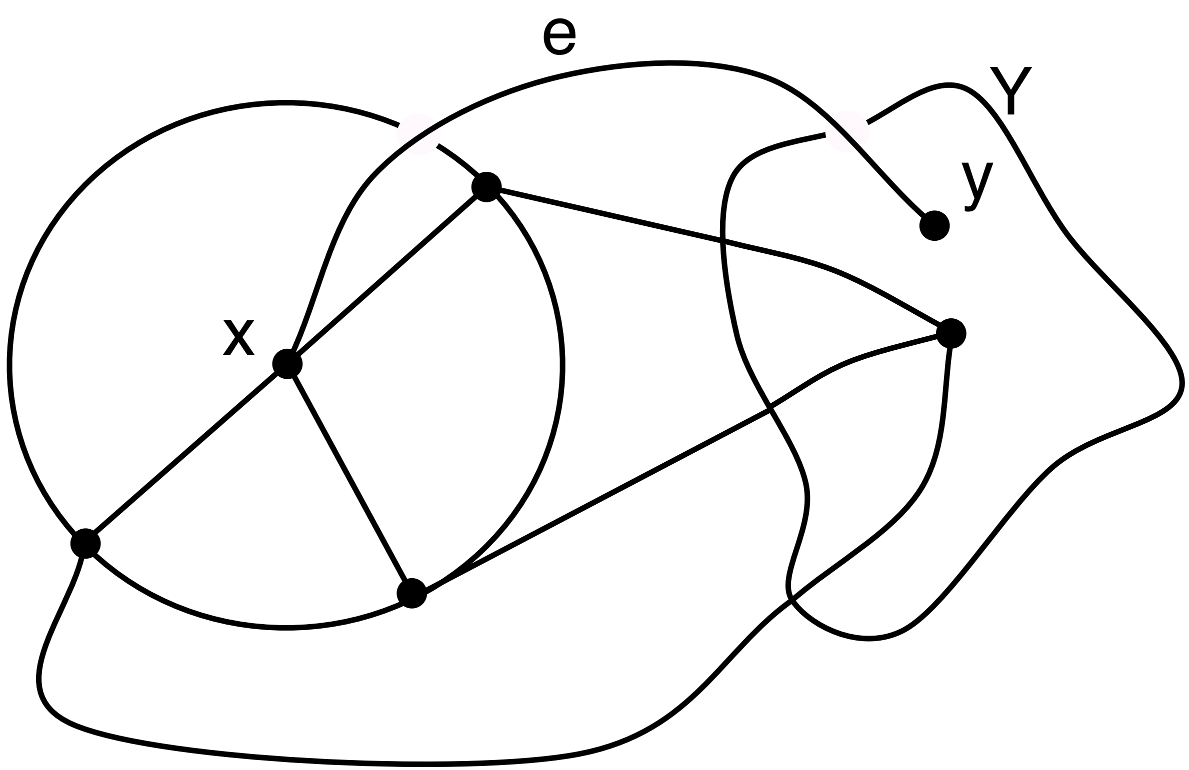
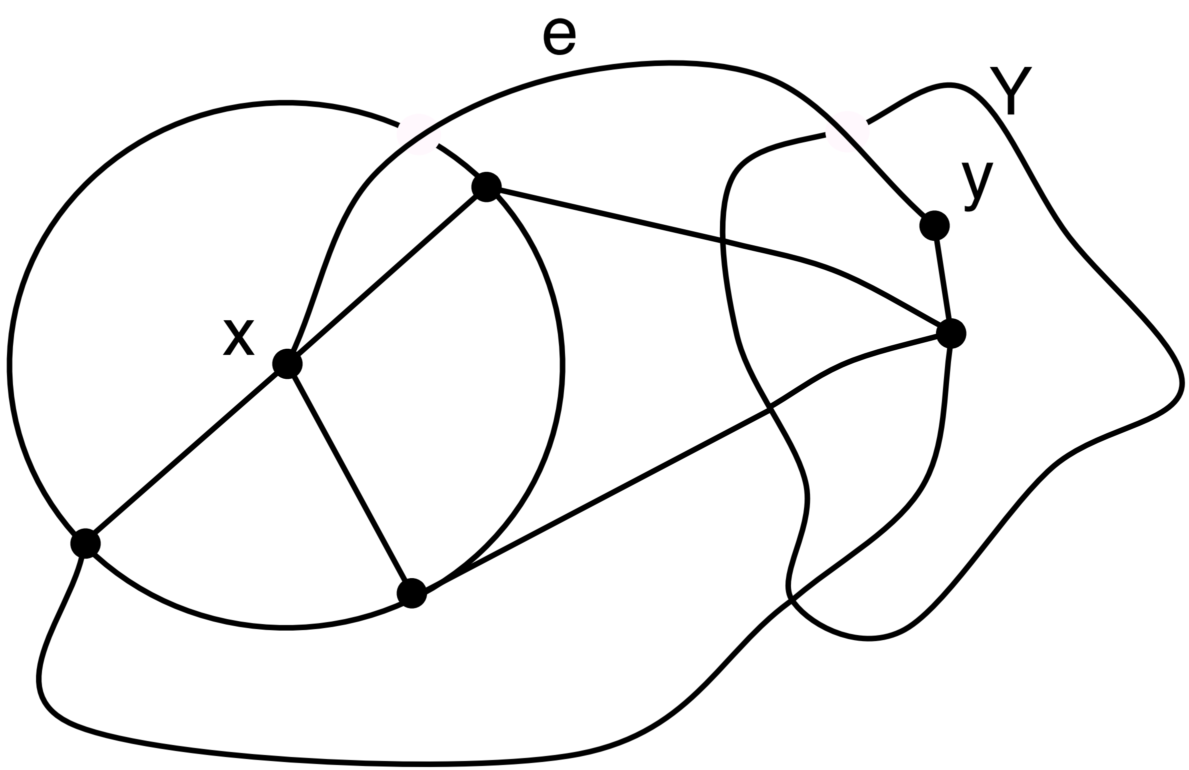


Suppose $\overline{Y}$ meets R in three or more endpoints of spokes.

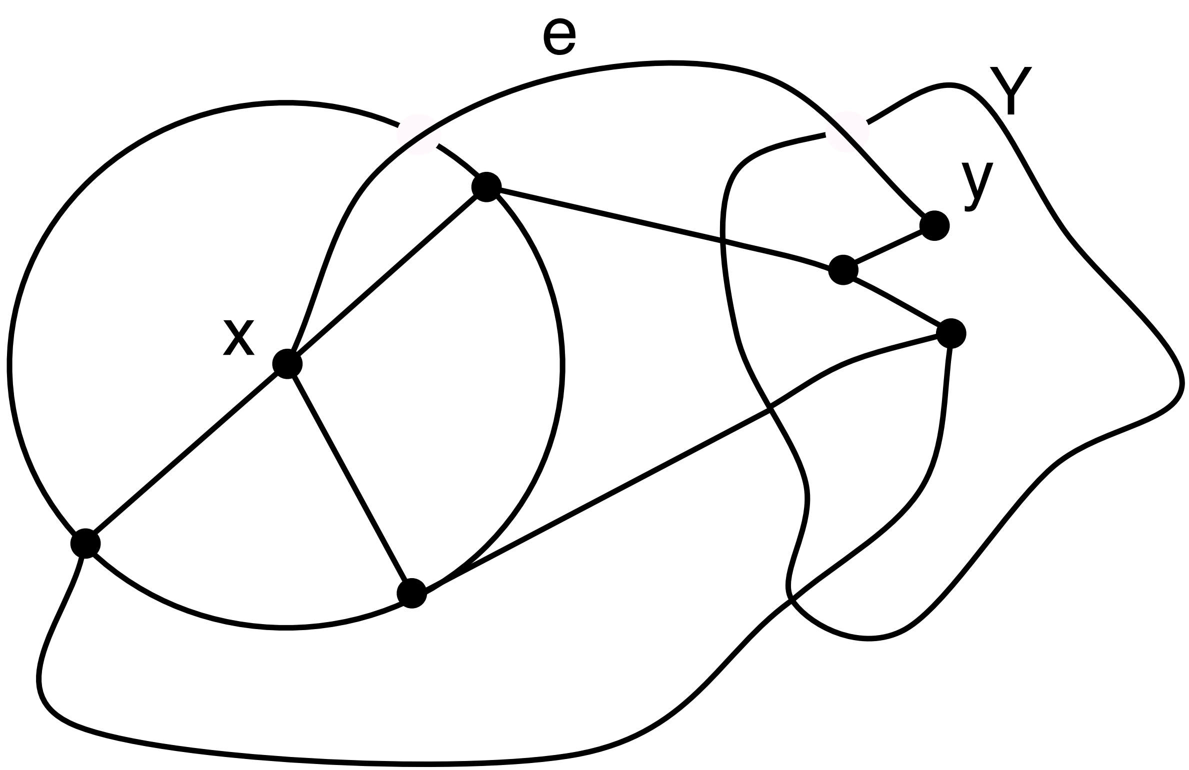


Suppose $\overline{Y}$ meets R in three or more endpoints of spokes.



case 3

Suppose $\overline{Y}$ meets R in three or more endpoints of spokes.



case 3

Suppose $\overline{Y}$ meets R in three or more endpoints of spokes.