Math 121 – Exam 2 Solutions

1. (10 pts) Determine whether the given statement is TRUE or FALSE. Briefly explain your reason for each answer.

   (a) $\dfrac{1}{a+bi} = a - bi$

   Solution: FALSE. $\dfrac{1}{a+bi} = \dfrac{1}{a+bi} \cdot \dfrac{a-bi}{a-bi} = \dfrac{a-bi}{a^2+b^2} \neq a - bi$

   (b) $\log(a \cdot b) = \log a \log b$

   Solution: FALSE. $\log(a \cdot b) = \log a + \log b \neq \log a \log b$

   (c) $a^{1/2} + b^{1/2} = (a+b)^{1/2}$

   Solution: FALSE. Counter-example: Let $a = 4$ and $b = 1$. Then $a^{1/2} + b^{1/2} = 4^{1/2} + 1^{1/2} = 3$ but $(a+b)^{1/2} = (4+1)^{1/2} = \sqrt{5}$.

   (d) $a - \ln e^a = 0$

   Solution: TRUE. $a - \ln e^a = a - a \ln e = a - a = 0$

   (e) $3^a \cdot 3^b = 3^{a+b}$

   Solution: TRUE. The exponentials on the left have the same base so when you multiply them, you can add the exponents.

2. (15 pts) For the rational function $f(x) = \dfrac{x-2}{3x-2}$,

   (a) Find the the x-intercept(s).

   Solution: $\boxed{x = 2}$ since it is a root of the numerator (but not the denominator)

   (b) Find the y-intercept (where the graph of $f(x)$ crosses the y-axis).

   Solution: $\boxed{y = 1}$ since $f(0) = \dfrac{0-2}{3(0)-2} = 1$

   (c) Find the vertical asymptote.

   Solution: $\boxed{x = \dfrac{2}{3}}$ since it is a root of the denominator (but not the numerator)

(d) Find the horizontal asymptote.

Solution: The degrees of the numerator and denominator are equal so the horizontal asymptote is $\boxed{y = \frac{1}{3}}$, the ratio of the coefficients of x in the numerator and denominator.

(e) Sketch the graph of $f(x)$, clearly indicating the above information on the graph.

3. (15 pts) Find a polynomial $f(x)$ of degree 4 whose roots are 0, 1, i, and $-i$ and satisfies $f(2) = 20$.

Solution: From the roots, we construct the general form of the polynomial:
$f(x) = ax(x-1)(x-i)(x+i)$ where a is a constant that is to be determined. Since $f(2) = 20$ we have:

$$\begin{aligned} f(2) = a(2)(2-1)(2-i)(2+i) &= 20 \\ a(2)(1)(4-i^2) &= 20 \\ a(2)(1)(4-(-1)) &= 20 \\ a(2)(1)(5) &= 20 \\ 10a &= 20 \\ a &= 2 \end{aligned}$$

So, $\boxed{f(x) = 2x(x-1)(x-i)(x+i)}$.

4. (15 pts) Solve: $x - 1 < 2x + 3 < 4 - x$.

Solution: We must break up the inequality into 2 inequalities:
(1) $x - 1 < 2x + 3$ AND (2) $2x + 3 < 4 - x$.
Solving (1):

$$\begin{aligned} x - 1 &< 2x + 3 \\ -x &< 4 \\ x &> -4 \end{aligned}$$

Solving (2):

$$\begin{aligned} 2x + 3 &< 4 - x \\ 3x &< 1 \\ x &< \frac{1}{3} \end{aligned}$$

So, the solution is $\boxed{-4 < x < \frac{1}{3}}$.

5. (15 pts) Find all roots, real and complex, of $x^3 - x^2 + 2 = 0$.

Solution: Let $f(x) = x^3 - x^2 + 2$. We first look for the rational roots of $f(x)$. We have $a_0 = \pm 1$, ± 2 and $a_3 = \pm 1$. So the possible rational roots are ± 1 and ± 2. Of these possibilities, only -1 works. That is, $f(-1) = 0$. So we know that $x + 1$ is a factor of $f(x)$. To get the other roots, we must divide $f(x)$ by $x + 1$:

$$
\begin{array}{r|l}
 & x^2 - 2x + 2 \\
\hline
x+1 & x^3 - x^2 \qquad + 2 \\
 & -x^3 - x^2 \\
\hline
 & -2x^2 \\
 & 2x^2 + 2x \\
\hline
 & 2x + 2 \\
 & -2x - 2 \\
\hline
 & 0
\end{array}
$$

Now we use the quadratic formula to find the roots of the quotient $q(x) = x^2 - 2x + 2$:

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} \\
&= \frac{2 \pm \sqrt{4 - 8}}{2} \\
&= \frac{2 \pm \sqrt{-4}}{2} \\
&= \frac{2 \pm 2i}{2} \\
x &= 1 \pm i
\end{aligned}
$$

Thus, the roots are $\boxed{-1, 1 + i, 1 - i}$.

6. (15 pts) Suppose you put \$1000 into a savings account with an interest rate of 3%, compounded monthly. How long will it take for the account balance to reach \$1500?

Solution: We start with the following formula for determining how much money is in an account $A(t)$ after t years with a starting balance of P and interest r compounded n times per year:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

Here, we have $A(t) = 1500$, $P = 1000$, $r = 0.03$, and $n = 12$ (compounded monthly).

Plugging these into the above equation and solving for t we have:

$$\begin{aligned}
1500 &= 1000\left(1+\frac{0.03}{12}\right)^{12t}\\
\frac{1500}{1000} &= (1+0.0025)^{12t}\\
1.5 &= (1.0025)^{12t}\\
\ln 1.5 &= \ln(1.0025)^{12t}\\
\ln 1.5 &= (12t)\ln 1.0025\\
t &= \frac{\ln 1.5}{12\ln 1.0025}\\
&\boxed{t = 13.53 \text{ years}}
\end{aligned}$$

7. (15 pts) Find all solutions to the equation: $2^{x-2} = 3^{2x+1}$.

Solution: To solve the equation we take the natural logarithm of both sides and then solve for x:

$$\begin{aligned}
2^{x-2} &= 3^{2x+1}\\
\ln 2^{x-2} &= \ln 3^{2x+1}\\
(x-2)\ln 2 &= (2x+1)\ln 3\\
(\ln 2)x - 2\ln 2 &= (2\ln 3)x + \ln 3\\
(\ln 2)x - (2\ln 3)x &= 2\ln 2 + \ln 3\\
x(\ln 2 - 2\ln 3) &= 2\ln 2 + \ln 3\\
&\boxed{x = \frac{2\ln 2 + \ln 3}{\ln 2 - 2\ln 3} = -1.65}
\end{aligned}$$