Exam 1 Solutions

1. (20 pts) Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

   (a) Write the row echelon form of A.

   (b) Write the reduced row echelon form of A.

   (c) Compute $\det A$ using whatever method you wish. If you use the 3×3 "trick", clearly indicate how you arrived at your answer.

   (d) Find all solutions to $A\mathbf{x} = \mathbf{b}$ using whatever method you wish.

Solution:

(a)

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -4 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -4 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow[R_3 \to (-1) \times R_3]{R_2 \to (-1) \times R_2} \boxed{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}}$$

(b)

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 \to R_2 - 4R_3]{R_1 \to R_1 - 2R_3} \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

(c)

$$\begin{aligned} \det A &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \\ &= a_{11}(-1)^{1+1} \det M_{11} + a_{12}(-1)^{1+2} \det M_{12} + a_{13}(-1)^{1+3} \det M_{13} \\ &= (1)(1)\begin{vmatrix} -1 & 0 \\ 1 & 3 \end{vmatrix} + (0)(-1)\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} + (2)(1)\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} \\ &= -3 + 0 + 4 \end{aligned}$$

$$\boxed{\det A = 1}$$

(d) Using the same row operations on $\mathbf{b}$ as those used to transform A to row reduced echelon form we get:

$$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Therefore, the solution is: $\boxed{x_1 = 1,\ x_2 = 1,\ x_3 = 0}$.

2. (15 pts) Let $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| x_1^2 - x_2^2 = 0 \right\}$. Is S a subspace of $\mathbb{R}^2$? Clearly show why or why not.

Solution: Let $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ be vectors in S. These certainly satisfy the condition: $x_1^2 - x_2^2 = 0$. Then $\mathbf{x} + \mathbf{y} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. However, $2^2 - 0^2 = 4 \neq 0$. Therefore, $\mathbf{x} + \mathbf{y} \notin S$. So $\boxed{S \text{ is not a subspace of } \mathbb{R}^2}$.

3. (10 pts) Use Cramer's Rule to solve the system of equations $A\mathbf{x} = \mathbf{b}$ where:

$$A = \begin{bmatrix} 2 & 3 \\ -4 & 5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}.$$

Solution:

$$x_1 = \frac{\det A_1}{\det A} = \frac{\begin{vmatrix} 6 & 3 \\ 7 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ -4 & 5 \end{vmatrix}} = \boxed{\frac{9}{22}}$$

$$x_2 = \frac{\det A_2}{\det A} = \frac{\begin{vmatrix} 2 & 6 \\ -4 & 5 \end{vmatrix}}{22} = \boxed{\frac{34}{22}}$$

4. (25 pts) Consider the matrix:

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 0 & -2 & -3 \end{bmatrix}.$$

(a) Find lower and upper triangular matrices L and U, respectively, such that $B = LU$.
(b) Determine $N(B)$, the nullspace of B. Write your answer in set notation.

Solution:

(a)

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 0 & -2 & -3 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & -3 \\ 0 & -2 & -3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \boxed{\begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & -3 \\ 0 & 0 & 0 \end{bmatrix} = U}$$

The elementary matrices associated with each row operation are:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Their inverses are:

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Since $E_2 E_1 B = U$, we have $B = (E_2 E_1)^{-1} U = E_1^{-1} E_2^{-1} U = LU$. Therefore, we have:

$$\boxed{L = E_1^{-1} E_2^{-2} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}$$

(b) To find the nullspace of B we must find all solutions to $B\mathbf{x} = \mathbf{0}$. To do this we will reduce B to row reduced echelon form. This process was already started above so we will continue the reduction of the U matrix:

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & -3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_2 \to (-1/2)\times R_2]{R_1 \to R_1 + \frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & -5/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Column 3 does not contain a pivot so x_3 is a free variable. Let $x_3 = \alpha$. Then from the row reduced echelon form of B we have:

$$x_1 = \frac{5}{2}\alpha$$
$$x_2 = -\frac{3}{2}\alpha$$

The nullspace of B is:

$$\boxed{N(B) = \left\{ \alpha \begin{bmatrix} 5/2 \\ -3/2 \end{bmatrix} \middle| \alpha \in \mathbb{R} \right\}}$$

5. (15 pts) Suppose $C \in \mathbb{R}^{n\times n}$ and $\det C = 1$. True or false?

(a) C is singular. FALSE

(b) $C\mathbf{x} = \mathbf{b}$ will not have a solution for all $\mathbf{b} \in \mathbb{R}^n$. FALSE

(c) The row reduced echelon form of C is the identity matrix. TRUE

(d) The determinant of C^T is not necessarily 1. FALSE

(e) The nullspace of C only contains the zero vector, $\mathbf{0}$. TRUE

6. (15 pts) Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(a) Write $\mathbf{b} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

(b) Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a spanning set for $\mathbb{R}^2$? Clearly explain why or why not.

(c) Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a spanning set for $\mathbb{R}^2$? Clearly explain why or why not.

Solution:

(a) $\boxed{\mathbf{b} = \mathbf{v}_2 - \mathbf{v}_1}$ since $\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(b) $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a spanning set for $\mathbb{R}^2$ if any vector $\mathbf{b}$ in $\mathbb{R}^2$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. That is, there must be a solution to $A\mathbf{x} = \mathbf{b}$, where $A = [\mathbf{v}_1\ \mathbf{v}_2]$, for all choices of $\mathbf{b}$. Since $\det A = -3 \neq 0$, A is invertible and $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution to the system no matter what $\mathbf{b}$ is. Therefore, $\boxed{\{\mathbf{v}_1, \mathbf{v}_2\} \text{ is a spanning set for } \mathbb{R}^2}$.

(c) Since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a spanning set for $\mathbb{R}^2$, $\boxed{\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \text{ is also a spanning set}}$ since we can take any $\mathbf{b}$ and form it as a linear combination of the given vectors as follows:

$$\mathbf{b} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + 0 \cdot \mathbf{v}_3$$

where $\mathbf{x} = [\alpha_1, \alpha_2]^T$ is the solution to the system in part (b).