Exam 1 Solutions

1. (20 pts) Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 3 & -1 & 1 \\ 2 & 0 & -3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

   (a) Row reduce the augmented matrix $[A|\mathbf{b}]$ until A is in row echelon form.

   (b) Compute $\det A$ using whatever method you wish. If you use the 3×3 "trick", clearly indicate how you arrived at your answer.

   (c) Find all solutions to $A\mathbf{x} = \mathbf{b}$ using whatever method you wish.

Solution:

(a)

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 3 & -1 & 1 & 2 \\ 2 & 0 & -3 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 2 & -3 & -1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & -4 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 \to (-1/4) \times R_3]{R_2 \to (1/2) \times R_2} \boxed{\left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1/2 & -1/2 \\ 0 & 0 & 1 & 0 \end{array}\right]}$$

(b) In the second to last set of row operations above, we have the following reduced form of A:

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -4 \end{bmatrix}$$

This matrix was obtained by using type III row operations which do not change the value of the determinant. Therefore, since the determinant of the above matrix is the product of the diagonal entries (since the matrix is upper triangular), we have $\boxed{\det A = (1)(2)(-4) = -8}$.

(c) From the row echelon form of $[A|\mathbf{b}]$ we have:

$$\begin{aligned} x_1 - x_2 &= 1 \\ x_2 + \frac{1}{2}x_3 &= -\frac{1}{2} \\ x_3 &= 0 \end{aligned}$$

The solution to this system is $\boxed{x_1 = \frac{1}{2}, x_2 = -\frac{1}{2}, x_3 = 0}$.

2. (15 pts) Let $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \, x_1 - |x_2| = 0 \right\}$. Is S a subspace of $\mathbb{R}^2$? Clearly explain why or why not.

Solution: $\mathbf{0} \in S$ since $x_1 = x_2 = 0$ satisfies the condition: $x_1 - |x_2| = 0$. However, $\alpha\mathbf{x}$ will not necessarily be in S. As an example, let $\mathbf{x} = [1, 1]^T$ and $\alpha = -1$. Certainly, $\mathbf{x} \in S$ since $1 - |1| = 0$. But $\alpha\mathbf{x} = [-1, -1]^T$ is not in S since $-1 - |-1| = -2 \neq 0$. Therefore, $\boxed{S \text{ is not a subspace of } \mathbb{R}^2}$.

3. (10 pts) Use Cramer's Rule to solve the system of equations $A\mathbf{x} = \mathbf{b}$ where:

$$A = \begin{bmatrix} 3 & 2 \\ -5 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Solution:

$$x_1 = \frac{\det A_1}{\det A} = \frac{\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ -5 & 4 \end{vmatrix}} = \frac{0}{22} = \boxed{0}$$

$$x_2 = \frac{\det A_2}{\det A} = \frac{\begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix}}{22} = \frac{11}{22} = \boxed{\frac{1}{2}}$$

4. (25 pts) Consider the matrix:

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 0 & -2 & -3 \end{bmatrix}.$$

B can be row reduced to an upper triangular matrix U as follows:

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 0 & -2 & -3 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -3 \\ 0 & -2 & -3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & 0 \end{bmatrix} = U$$

(a) Find the lower triangular matrix L such that $B = LU$.

(b) Determine $N(B)$, the nullspace of B. (Hint: The matrix U might be helpful.)

Solution:

(a) $\boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}$

(b) The nullspace of B is the set of all solutions to $B\mathbf{x} = \mathbf{0}$. Further row reducing U, we get:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to (-1/2) \times R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Since Column 3 does not contain a pivot, x_3 is a free variable. Let $x_3 = \alpha$. Then we have $x_1 = \alpha/2$ and $x_2 = -3\alpha/2$. The nullspace of B is then:

$$\boxed{N(B) = \left\{ \alpha \begin{bmatrix} 1/2 \\ -3/2 \\ 1 \end{bmatrix} \middle| \alpha \in \mathbb{R} \right\}}$$

5. (15 pts) Suppose $C \in \mathbb{R}^{n \times n}$ and $\det C = 2$. True or false? No justification is needed.

(a) C is nonsingular. TRUE

(b) $C\mathbf{x} = \mathbf{b}$ will not have a solution for all $\mathbf{b} \in \mathbb{R}^n$. FALSE

(c) The reduced row echelon form of C is the identity matrix. TRUE

(d) $\det C^{-1} = 2$. FALSE

(e) The nullspace of C contains only the zero vector, $\mathbf{0}$. TRUE

6. (15 pts) Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(a) Write $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

(b) Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a spanning set for $\mathbb{R}^2$? Clearly explain why or why not.

(c) Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a spanning set for $\mathbb{R}^2$? Clearly explain why or why not.

Solution:

(a) $\boxed{\mathbf{b} = \mathbf{v}_2 - \mathbf{v}_1}$

(b) $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a spanning set for $\mathbb{R}^2$ if any vector $\mathbf{b}$ in $\mathbb{R}^2$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. That is, there must be a solution to $A\mathbf{x} = \mathbf{b}$, where $A = [\mathbf{v}_1 \ \mathbf{v}_2]$, for all choices of $\mathbf{b}$. Since $\det A = 1 \neq 0$, A is invertible and $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution to the system no matter what $\mathbf{b}$ is. Therefore, $\boxed{\{\mathbf{v}_1, \mathbf{v}_2\} \text{ is a spanning set for } \mathbb{R}^2}$.

(c) Since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a spanning set for $\mathbb{R}^2$, $\boxed{\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \text{ is also a spanning set}}$ since we can take any $\mathbf{b}$ and form it as a linear combination of the given vectors as follows:

$$\mathbf{b} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + 0 \cdot \mathbf{v}_3$$

where $\mathbf{x} = [\alpha_1, \alpha_2]^T$ is the solution to the system in part (b).