Exam 2 Solutions

1. (15 pts) Consider the following vectors:

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{a}_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \ \mathbf{a}_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

   (a) Are the vectors linearly independent? Clearly explain why or why not.

   (b) Let A be the matrix whose columns are $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$, i.e. $A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3]$. What is the rank of A?

Solution:

   (a) Let's make these vectors the columns of a matrix A as follows:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 4 \end{bmatrix}$$

   We look for all solutions to the system of equations $A\mathbf{x} = \mathbf{0}$. To do this, we find the row reduced echelon form of A:

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \end{bmatrix}$$

   Since the third column does not contain a pivot, x_3 will be a free variable. Therefore, there will be infinite solutions to $A\mathbf{x} = \mathbf{0}$ which means $\mathbf{x} = \mathbf{0}$ is not the only solution. Thus, $\boxed{\text{the vectors are linearly dependent}}$.

   (b) Using the row reduced echelon form of A above, we find that there are 2 non-zero rows (2 pivots). Therefore, $\boxed{\text{the rank of } A \text{ is } 2}$.

2. (15 pts) Let $S = \text{Span}\,\{1, \sin x, x\}$ be a subspace of $C[-1, 1]$.

   (a) Find a basis for S, clearly explaining why the functions you chose constitute a basis.

   (b) Determine $\dim S$.

Solution:

   (a) The functions 1, $\sin x$, and x certainly form a spanning set for S. But are they linearly independent? We'll use the Wronskian to check this:

$$W(x) = \begin{vmatrix} 1 & \sin x & x \\ 0 & \cos x & 1 \\ 0 & -\sin x & 0 \end{vmatrix}$$

$$W(x) = 1 \cdot \begin{vmatrix} \cos x & 1 \\ -\sin x & 0 \end{vmatrix}$$

$$W(x) = \sin x$$

   Since $W(1) = \sin 1 \neq 0$, the functions are linearly independent. Therefore, the functions $\boxed{1, \sin x, \text{ and } x \text{ form a basis for } S}$.

   (b) Since there are 3 functions in the basis, $\boxed{\text{the dimension of } S \text{ is } 3}$.

3. (15 pts) Consider the following mapping from $\mathbb{R}^2$ into $\mathbb{R}^2$:

$$L(\mathbf{x}) = \begin{bmatrix} 0 \\ x_1 - x_2 \end{bmatrix}$$

Show that L is a linear operator on $\mathbb{R}^2$.

To show that L is a linear operator, we must show that (1) $L(\mathbf{x}+\mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y})$ and (2) $L(\alpha\mathbf{x}) = \alpha L(\mathbf{x})$. Let's start with (1):

$$\begin{aligned} L(\mathbf{x}+\mathbf{y}) &= L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) \\ &= L\left(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} 0 \\ (x_1 + y_1) - (x_2 + y_2) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ x_1 - x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ y_1 - y_2 \end{bmatrix} \\ &\boxed{L(\mathbf{x}+\mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y})} \end{aligned}$$

Now let's show (2):

$$\begin{aligned} L(\alpha\mathbf{x}) &= L\left(\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} 0 \\ \alpha x_1 - \alpha x_2 \end{bmatrix} \\ &= \alpha \begin{bmatrix} 0 \\ x_1 - x_2 \end{bmatrix} \\ &\boxed{L(\alpha\mathbf{x}) = \alpha L(\mathbf{x})} \end{aligned}$$

4. (20 pts) Consider the following linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^3$:

$$L(\mathbf{x}) = \begin{bmatrix} x_1 \\ x_1 - x_2 \\ x_1 - x_3 \end{bmatrix}$$

Find a matrix representation of L with respect to each basis below:

(a) the standard basis

(b) the basis vectors $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ where

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Solution:

(a) We construct the columns of the matrix by plugging the standard basis vectors into the mapping:

$$L(\mathbf{e}_1) = L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
$$L(\mathbf{e}_2) = L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$
$$L(\mathbf{e}_3) = L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Therefore, the matrix representation of L with respect to the standard basis is:

$$\boxed{A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}}$$

(b) We start by plugging the **b** basis vectors into the mapping:

$$L(\mathbf{b}_1) = L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
$$L(\mathbf{b}_2) = L\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$L(\mathbf{b}_3) = L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

We must now write these vectors as linear combinations of the **b** basis vectors. This can be done by inspection as follows:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0 \cdot \mathbf{b}_1 + 1 \cdot \mathbf{b}_2 + 0 \cdot \mathbf{b}_3$$
$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \cdot \mathbf{b}_1 + 0 \cdot \mathbf{b}_2 + 0 \cdot \mathbf{b}_3$$
$$\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = 0 \cdot \mathbf{b}_1 + 0 \cdot \mathbf{b}_2 - 1 \cdot \mathbf{b}_3$$

The columns of the matrix representation are made up of the coefficients in the linear combinations above:

$$\boxed{B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}}$$

5. (15 pts) Let S be the subspace spanned by the following vectors:

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Find a basis for $S^\perp$, the orthogonal complement of S.

Solution: We start by defining $A^T = [\mathbf{w}_1\ \mathbf{w}_2\ \mathbf{w}_3]$. We know that $S = R(A^T)$ and, thus, $S^\perp = R(A^T)^\perp = N(A)$. Therefore, to find a basis for $S^\perp$ we must find a basis for $N(A)$. The matrix A row reduces as follows:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow \text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since columns 3 does not contain a pivot, x_3 will be a free variable when we consider the set of solutions to $A\mathbf{x} = \mathbf{0}$. Letting $x_3 = \alpha$ we then have $x_1 = -x_3 = -\alpha$ and $x_2 = 0$. Therefore, the nullspace of A is:

$$N(A) = \text{Span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and $\boxed{\text{a basis for } S^\perp \text{ is the vector } [-1\ 0\ 1]^T}$.

6. (20 pts) Suppose we have the following data:

x	-1	0	1	2
y	0	-1	1	3

(a) Find a function of the form $y = a + bx^2$ that best fits the data in the least squares sense.

(b) Sketch the points and the best fit function on the graph provided.

Solution:

(a) We start by plugging the data points into the function $y = a + bx^2$ to form a system of equations:

$$\begin{aligned} 0 &= a + b(-1)^2 \\ -1 &= a + b(0)^2 \\ 1 &= a + b(1)^2 \\ 3 &= a + b(2)^2 \end{aligned}$$

The system in matrix-vector form is:

$$A\mathbf{x} = \mathbf{b}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 3 \end{bmatrix}$$

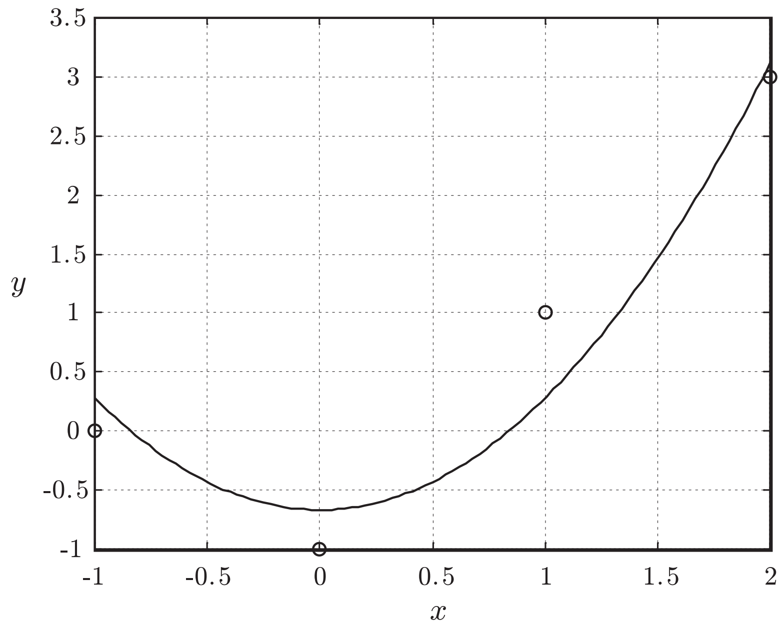
The least squares solution is $\mathbf{x} = (A^TA)^{-1}A^T\mathbf{b}$. First, we have:

$$A^TA = \begin{bmatrix} 4 & 6 \\ 6 & 18 \end{bmatrix},\ (A^TA)^{-1} = \frac{1}{36}\begin{bmatrix} 18 & -6 \\ -6 & 4 \end{bmatrix},\ A^T\mathbf{b} = \begin{bmatrix} 3 \\ 13 \end{bmatrix}$$

Then we have:

$$\mathbf{x} = \frac{1}{36}\begin{bmatrix} 18 & -6 \\ -6 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 13 \end{bmatrix} = \frac{1}{36}\left[\ -24//34\ \right]$$

Therefore, the function that best fits the data is $\boxed{y = -\frac{24}{36} + \frac{34}{36}x^2 = -\frac{2}{3} + \frac{17}{18}x^2}$.



(b)