Math 310 Final Exam Solutions

1. (20 pts) Consider the system of equations $A\mathbf{x} = \mathbf{b}$ where:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

   (a) Compute $\det A$. Is A singular or nonsingular?

   (b) Compute A^{-1}, if possible.

   (c) Write the row reduced echelon form of A.

   (d) Find all solutions to the system $A\mathbf{x} = \mathbf{b}$.

   Solution:

   (a) $\det A = 2 \Rightarrow A$ is nonsingular

   (b) $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$

   (c) $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   (d) $x_1 = 1$, $x_2 = 1$, $x_3 = 0$

2. (10 pts) Consider the following set S:

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \ x_1^2 - x_2^2 = 0 \right\}$$

   Is S a subspace of the vector space $\mathbb{R}^2$? Clearly explain why or why not.

   Solution: S is not a subspace of $\mathbb{R}^2$ because it is not closed under addition. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ be in S. Then we have $x_1^2 - x_2^2 = 0$ and $y_1^2 - y_2^2 = 0$. The sum $\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$ is not in S because:

$$(x_1 + y_1)^2 - (x_2 + y_2)^2 = 2(x_1 y_1 - x_2 y_2) \neq 0 \text{ for all possible } x_1, x_2, y_1, y_2$$

3. (20 pts) Consider the following matrix A:

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 0 \end{bmatrix}$$

   (a) Find the nullspace of A.

   (b) Do the columns of A form a spanning set for $\mathbb{R}^2$? Clearly explain why or why not.

   Solution:

(a) The row reduced echelon form of A is:

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

There is no pivot in the third column. Therefore, x_3 is a free variable. Let $x_3 = \alpha$. Then we have $x_1 + x_3 = 0$ and $x_2 - 2x_3 = 0$ which give us $x_1 = -\alpha$ and $x_2 = 2\alpha$. The nullspace of A is:

$$\boxed{N(A) = \left\{ \begin{bmatrix} -\alpha \\ 2\alpha \\ \alpha \end{bmatrix} \middle| \alpha \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}}$$

(b) The columns of A form a spanning set for $\mathbb{R}^2$ because there is a solution to $A\mathbf{x} = \mathbf{b}$ for every $\mathbf{b} \in \mathbb{R}^2$ (there are no zero rows in the row reduced echelon form of A).

4. (20 pts) Do the vectors below form a basis for $\mathbb{R}^3$? If so, explain. If not, remove as many vectors as you need to form a basis and show that the resulting set of vectors form a basis for $\mathbb{R}^3$.

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{x}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

Solution: The vectors do not form a basis for $\mathbb{R}^3$ because dim $\mathbb{R}^3 = 3$ so there can only be 3 vectors in any basis for $\mathbb{R}^3$. If we remove $\mathbf{x}_1$, then consider the matrix X whose columns are $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4$:

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Since $\det X = 1$, X is invertible and there is only one solution to $A\mathbf{x} = \mathbf{b}$ for every $\mathbf{b} \in \mathbb{R}^3$. Therefore, the columns are LI and form a spanning set for $\mathbb{R}^3$. Thus, they form a basis for $\mathbb{R}^3$.

5. (30 pts) Consider the following mapping $L: \mathbb{R}^2 \to \mathbb{R}^3$:

$$L(\mathbf{x}) = \begin{bmatrix} 2x_1 \\ -x_2 \\ x_1 + x_2 \end{bmatrix}$$

(a) Show that L is a linear transformation.

(b) Find a matrix representation for L using the standard basis for $\mathbb{R}^3$ and the following basis vectors for $\mathbb{R}^2$:

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Solution:

(a) Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Then,

$$L(\mathbf{x}+\mathbf{y}) = L\left(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}\right) = \begin{bmatrix} 2(x_1 + y_1) \\ -(x_2 + y_2) \\ (x_1 + y_1) + (x_2 + y_2) \end{bmatrix} = \begin{bmatrix} 2x_1 \\ -x_2 \\ x_1 + x_2 \end{bmatrix} + \begin{bmatrix} 2y_1 \\ -y_2 \\ y_1 + y_2 \end{bmatrix} = L(\mathbf{x}) + L(\mathbf{y})$$

$$L(\alpha\mathbf{x}) = L\left(\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix}\right) = \begin{bmatrix} 2\alpha x_1 \\ -\alpha x_2 \\ \alpha x_1 + \alpha x_2 \end{bmatrix} = \alpha \begin{bmatrix} 2x_1 \\ -x_2 \\ x_1 + x_2 \end{bmatrix} = \alpha L(\mathbf{x})$$

(b)

$$L(\mathbf{u}_1) = L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$L(\mathbf{u}_2) = L\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

The matrix representation of L is then:

$$\boxed{A = \begin{bmatrix} 2 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

6. (15 pts) Let $Y = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$ where:

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Find $Y^\perp$, the orthogonal complement of Y.

Solution: We use the fact that $Y^\perp = R(A^T)^\perp = N(A)$ where $A^T = [\ \mathbf{x}_1 \quad \mathbf{x}_2\ ]$. The matrix A is then the same as in Problem 3. Since we already found the nullspace of A in Problem 3, the answer is:

$$\boxed{Y^\perp = \text{Span}\left\{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}\right\}}$$

7. (15 pts) Use the Gram-Schmidt method to find an orthonormal basis for $\mathbb{R}^3$ from the basis:

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Solution:

$$\boxed{\mathbf{u}_1 = \frac{\mathbf{x}_1}{||\mathbf{x}_1||} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

$$\mathbf{p}_1 = \langle \mathbf{x}_2, \mathbf{u}_1 \rangle \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{\mathbf{u}_2 = \frac{\mathbf{x}_2 - \mathbf{p}_1}{||\mathbf{x}_2 - \mathbf{p}_1||} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

$$\mathbf{p}_2 = \langle \mathbf{x}_3, \mathbf{u}_1 \rangle \mathbf{u}_1 + \langle \mathbf{x}_3, \mathbf{u}_2 \rangle \mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{\mathbf{u}_3 = \frac{\mathbf{x}_3 - \mathbf{p}_2}{||\mathbf{x}_3 - \mathbf{p}_2||} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$$

8. (20 pts) Find a matrix X and a diagonal matrix D such that $A = XDX^{-1}$ where

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Solution: The eigenvalues of A are found as follows:

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \begin{vmatrix} 1-\lambda & 0 & -2 \\ 0 & 3-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} &= 0 \\ (3-\lambda)\begin{vmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} &= 0 \\ (3-\lambda)[(1-\lambda)^2 - (-2)^2] &= 0 \\ (3-\lambda)(1 - 2\lambda + \lambda^2 - 4) &= 0 \\ (3-\lambda)(\lambda^2 - 2\lambda - 3) &= 0 \\ (3-\lambda)(\lambda - 3)(\lambda + 1) &= 0 \\ \lambda &= -1,\ \lambda = 3 \text{ (repeated)} \end{aligned}$$

Plugging $\lambda = -1$ into $(A - \lambda I)\mathbf{x} = \mathbf{0}$ we get:

$$\begin{gathered} (A + I)\mathbf{x} = \mathbf{0} \\ \begin{bmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{gathered}$$

The first and third equations tell us $x_1 - x_3 = 0$ and the second equation tells us $x_2 = 0$. Since x_3 is a free variable, let $x_3 = \alpha$. Then we have $x_1 = \alpha$. Setting $\alpha = 1$ we get the eigenvector:

$$\lambda_1 = -1,\ \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Plugging $\lambda = 3$ into $(A - \lambda I)\mathbf{x} = \mathbf{0}$ we get:

$$\begin{gathered} (A - 3I)\mathbf{x} = \mathbf{0} \\ \begin{bmatrix} -2 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{gathered}$$

The first and third equations tell us $x_1 + x_3 = 0$ and the second equation tells us $0 = 0$. Since both x_2 and x_3 are free variables, let $x_2 = \alpha$ and $x_3 = \beta$. Then we have $x_1 = -\beta$. The set of solutions is then:

$$\mathbf{x} = \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Letting $\alpha = \beta = 1$, we get the eigenvectors:

$$\lambda_{2,3} = 3,\ \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{x}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

The matrix X has the eigenvectors as its columns and the diagonal matrix D has the corresponding eigenvalues along the main diagonal:

$$\boxed{X = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}}$$

9. (20 pts) Find the solution to the system of first order ODEs:

$$\frac{dy_1}{dt} = y_1 - 4y_2, \quad y_1(0) = 3$$
$$\frac{dy_2}{dt} = -y_2, \quad y_2(0) = 2$$

Solution: Writing this system in matrix-vector form we have:

$$\mathbf{y}' = A\mathbf{y}$$
$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Since A is upper triangular, the eigenvalues are on the main diagonal: $\lambda = 1, -1$. Plugging $\lambda = 1$ into $(A - \lambda I)\mathbf{x} = \mathbf{0}$ we get:

$$(A - I)\mathbf{x} = \mathbf{0}$$
$$\begin{bmatrix} 0 & -4 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Both equations tell us that $x_2 = 0$. However, x_1 is free so we let $x_1 = \alpha$. Setting $\alpha = 1$ we get the eigenvector:

$$\lambda_1 = 1, \ \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Plugging $\lambda = -1$ into $(A - \lambda I)\mathbf{x} = \mathbf{0}$ we get:

$$(A + I)\mathbf{x} = \mathbf{0}$$
$$\begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The first equation tells us that $x_1 - 2x_2 = 0$. Since x_2 is free we let $x_2 = \alpha$, which gives us $x_1 = 2\alpha$. Setting $\alpha = 1$ we get the eigenvector:

$$\lambda_2 = -1, \ \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The general solution to the system is:

$$\mathbf{y}(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2$$
$$\mathbf{y}(t) = c_1 e^{t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Plugging in the initial conditions we get:

$$\mathbf{y}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The solution to this system of algebraic equations is $c_1 = -1$ and $c_2 = 2$. Therefore, the solution is:

$$\boxed{\mathbf{y}(t) = -e^{t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

10. (a) (10 pts) Let $\mathbf{z} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} i \\ 2-i \end{bmatrix}$. Compute $||\mathbf{z}||$, $\langle \mathbf{z}, \mathbf{w} \rangle$, and $\langle \mathbf{w}, \mathbf{z} \rangle$.

(b) (20 pts) Consider the following matrix:

$$M = \begin{bmatrix} 2 & i & 0 \\ -i & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Show that M is Hermitian and find a unitary matrix U that diagonalizes M.

Solution:

(a)

$$||\mathbf{z}|| = \sqrt{\mathbf{z}^H \mathbf{z}} = \sqrt{\begin{bmatrix} 1-i & 1 \end{bmatrix} \begin{bmatrix} 1+i \\ 1 \end{bmatrix}} = \sqrt{(1-i)(1+i) + (1)(1)} = \sqrt{1 - i^2 + 1} = \boxed{\sqrt{3}}$$

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^H \mathbf{z} = \begin{bmatrix} -i & 2+i \end{bmatrix} \begin{bmatrix} 1+i \\ 1 \end{bmatrix} = (-i)(1+i) + (2+i)(1) = -i - i^2 + 2 + i = \boxed{3}$$

$$\langle \mathbf{w}, \mathbf{z} \rangle = \overline{\langle \mathbf{z}, \mathbf{w} \rangle} = \boxed{3}$$

(b) To show that M is Hermitian, we must show that $M = M^H$:

$$M^H = \overline{M}^T = \begin{bmatrix} 2 & -i & 0 \\ i & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & i & 0 \\ -i & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = M$$

To find the unitary matrix that diagonalizes M we must find the eigenvalues and eigenvectors for M. The eigenvalues are found as follows:

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 2-\lambda & i & 0 \\ -i & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda) \begin{vmatrix} 2-\lambda & i \\ -i & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)[(2-\lambda)^2 - (i)(-i)] = 0$$

$$(2-\lambda)(4 - 4\lambda + \lambda^2 - 1) = 0$$

$$(2-\lambda)(\lambda^2 - 4\lambda + 3) = 0$$

$$(2-\lambda)(\lambda - 1)(\lambda - 3) = 0$$

$$\lambda = 1, 2, 3$$

The corresponding eigenvectors are:

$$\lambda_1 = 1,\ \mathbf{x}_1 = \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix};\ \lambda_2 = 2,\ \mathbf{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix};\ \lambda_3 = 3,\ \mathbf{x}_3 = \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}$$

Since M is Hermitian, the eigenvectors are orthogonal. Therefore, in order to construct the unitary matrix U we simply need to normalize the eigenvectors. The norms of both $\mathbf{x}_1$ and $\mathbf{x}_3$ are $\sqrt{2}$. Therefore, the unitary matrix is:

$$\boxed{U = \begin{bmatrix} -\frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}}$$

Bonus: (10 pts) Consider the set S which consists of all cubic polynomials $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ that satisfy the equation $p''(0) + 4p(0) = 0$. That is,

$$S = \{p(t) \mid p(t) \in P_4,\ p''(0) + 4p(0) = 0\}$$

(a) Show that S is a subspace of P_4.

(b) Find a basis for S.

Solution:

1. (a) $p(t) = 0$ is certainly in S because it satisfies the condition:

$$p''(0) + 4p(0) = 0 + 4(0) = 0$$

(b) Let $p(t), q(t) \in S$. Therefore, we have:

$$\begin{aligned} p''(0) + 4p(0) &= 0 \\ q''(0) + 4q(0) &= 0 \end{aligned}$$

Let $r(t) = p(t) + q(t)$. Then we have $r''(t) = p''(t) + q''(t)$ and:

$$\begin{aligned} r''0 + 4r(0) &= p''(0) + q''(0) + 4(p(0) + q(0)) \\ &= p''(0) + 4p(0) + q''(0) + 4q(0) \\ &= 0 + 0 \\ r''0 + 4r(0) &= 0 \end{aligned}$$

Therefore, $r(t) = p(t) + q(t) \in S$.

(c) Let $p(t) \in S$ and $\alpha \in \mathbb{R}$. Since $p(t) \in S$ we have:

$$p''(0) + 4p(0) = 0$$

Let $r(t) = \alpha p(t)$. Then we have $r''(t) = \alpha p''(t)$ and:

$$\begin{aligned} r''0 + 4r(0) &= \alpha p''(0) + 4\alpha p(0) \\ &= \alpha(p''(0) + 4p(0)) \\ &= \alpha(0) \\ r''0 + 4r(0) &= 0 \end{aligned}$$

Therefore, $r(t) = \alpha p(t) \in S$.

Since the above three conditions are satisfied, S is a subspace of P_4.

2. Let $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$. Then $p''(t) = 2a_2 + 6a_3 t$. The condition then tells us that:

$$\begin{aligned} p''(0) + 4p(0) &= 0 \\ 2a_2 + 6a_3(0) + 4(a_0 + a_1(0) + a_2(0)^2 + a_3(0)^3) &= 0 \\ 2a_2 + 4a_0 &= 0 \\ a_2 &= -2a_0 \end{aligned}$$

Therefore, the set S can be rewritten as follows:

$$\begin{aligned} S &= \left\{a_0 + a_1 t - 2a_0 t^2 + a_3 t^3 | a_0, a_1, a_3 \in \mathbb{R}\right\} \\ S &= \left\{a_0(1 - 2t^2) + a_1 t + a_3 t^3 | a_0, a_1, a_3 \in \mathbb{R}\right\} \\ S &= \text{Span}\{1 - 2t^2, t, t^3\} \end{aligned}$$

The functions $1 - 2t^2, t, t^3$ are LI and span S. Therefore, they form a basis for S.