Math 310 Homework 10 Solutions

Chapter 5, Section 3

3. (a)

$$A^TA = \begin{bmatrix} 6 & 12 \\ 12 & 24 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$$

The matrix A^TA is not invertible since $\det A^TA = 0$. However, there is still a least squares solution. In fact, there are infinite least squares solutions here. We find them by solving the system:

$$A^TA\mathbf{x} = A^T\mathbf{b}$$

$$\begin{bmatrix} 6 & 12 \\ 12 & 24 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$$

Both equations in the system give us the same equation:

$$x_1 + 2x_2 = 1$$

Let $x_2 = \alpha$. Then $x_1 = 1 - 2x_2 = 1 - 2\alpha$. Therefore, the least squares solutions are:

$$\mathbf{x} = \begin{bmatrix} 1-2\alpha \\ \alpha \end{bmatrix}, \quad \alpha \in \mathbb{R}$$

(b)

$$A^TA = \begin{bmatrix} 3 & 0 & 6 \\ 0 & 14 & 14 \\ 6 & 14 & 26 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} 6 \\ 14 \\ 26 \end{bmatrix}$$

The matrix A^TA is not invertible since $\det A^TA = 0$. However, there is still a least squares solution. In fact, there are infinite least squares solutions here just as in part (a). We find them by solving the system:

$$A^TA\mathbf{x} = A^T\mathbf{b}$$

$$\begin{bmatrix} 3 & 0 & 6 \\ 0 & 14 & 14 \\ 6 & 14 & 26 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 26 \end{bmatrix}$$

The row reduced echelon form of the augmented matrix $[A^TA|A^T\mathbf{b}]$ is:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The third column does not contain a pivot. Therefore, x_3 is a free variable. Let $x_3 = \alpha$. Then $x_2 = 1 - x_3 = 1 - \alpha$ and $x_1 = 2 - 2x_3 = 2 - 2\alpha$. Therefore, the least squares solutions are:

$$\boxed{\mathbf{x} = \begin{bmatrix} 2-2\alpha \\ 1-\alpha \\ \alpha \end{bmatrix}, \quad \alpha \in \mathbb{R}}$$

5. (a) A linear function is of the form $y = mx + b$. Plugging in the data, we get the following system of equations:

$$\begin{aligned} 0 &= -1m + b \\ 1 &= 0m + b \\ 3 &= 1m + b \\ 9 &= 2m + b \end{aligned}$$

In matrix-vector format, the system is:

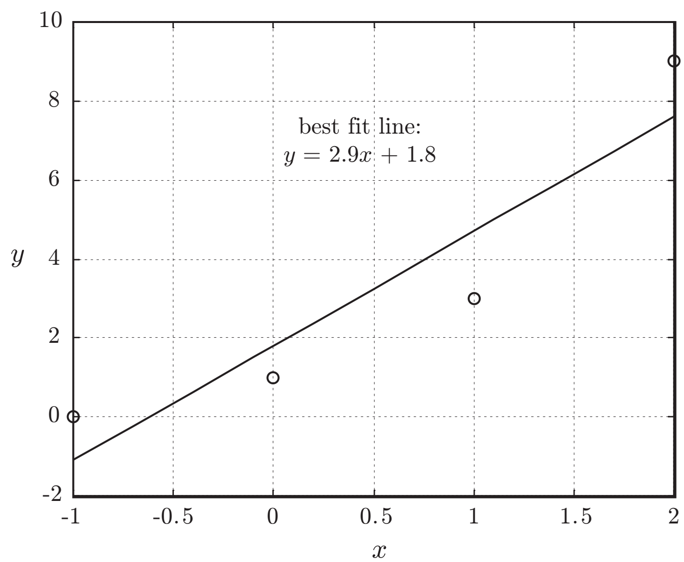
$$\underbrace{\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ b \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 3 \\ 9 \end{bmatrix}}_{\mathbf{b}}$$

The least squares solution is:

$$\mathbf{x} = (A^T A)^{-1} A^T \mathbf{b} = \begin{bmatrix} 2.9 \\ 1.8 \end{bmatrix}$$

The equation of the best fit line is:

$$\boxed{y = 2.9x + 1.8}$$



(b)

Chapter 5, Section 4

4. (a) $\langle A, B \rangle = (1)(-4) + (2)(1) + (2)(1) + (1)(-3) + (0)(3) + (2)(2) + (3)(1) + (1)(-2) + (1)(-2) = \boxed{0}$

(b) $||A||_F = \sqrt{1^2 + 2^2 + 2^2 + 1^2 + 0^2 + 2^2 + 3^2 + 1^2 + 1^2} = \boxed{5}$

(c) $||B||_F = \sqrt{(-4)^2 + 1^2 + 1^2 + (-3)^2 + 3^2 + 2^2 + 1^2 + (-2)^2 + (-2)^2} = \boxed{7}$

(d) $||A + B||_F = \sqrt{(-3)^2 + 3^2 + 3^2 + (-2)^2 + 3^2 + 4^2 + 4^2 + (-1)^2 + (-1)^2} = \boxed{\sqrt{74}}$

7. (a) $\langle e^x, e^{-x}\rangle = \int_0^1 e^x e^{-x}\,dx = \int_0^1 dx = \boxed{1}$

(b) $\langle x, \sin \pi x\rangle = \int_0^1 x \sin \pi x\,dx = \left[\frac{\sin \pi x}{\pi^2} - \frac{x\cos \pi x}{\pi}\right]_0^1 = \boxed{\frac{1}{\pi}}$

(c) $\langle x^2, x^3\rangle = \int_0^1 x^2 x^3\,dx = \int_0^1 x^5\,dx = \boxed{\frac{1}{6}}$