Math 310 Homework 11 Solutions

Chapter 5, Section 5

1. (a) orthonormal basis
   (b) not an orthonormal basis - the vectors are unit vectors but they are not orthogonal
   (c) not an orthonormal basis - neither vector is a unit vector
   (d) orthonormal basis

2. (a) To show that the given vectors form an orthonormal basis we must show (1) the vectors are unit vectors and (2) the vectors are orthogonal. Let's show (1):

$$||\mathbf{u}_1|| = \sqrt{\left(\frac{1}{3\sqrt{2}}\right)^2 + \left(\frac{1}{3\sqrt{2}}\right)^2 + \left(-\frac{4}{3\sqrt{2}}\right)^2} = \sqrt{\frac{1}{18} + \frac{1}{18} + \frac{16}{18}} = \sqrt{1} = 1$$

$$||\mathbf{u}_2|| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}} = \sqrt{1} = 1$$

$$||\mathbf{u}_3|| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 + (0)^2} = \sqrt{\frac{1}{2} + \frac{1}{2} + 0} = \sqrt{1} = 1$$

Now let's show (2):

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \left(\frac{1}{3\sqrt{2}}\right)\left(\frac{2}{3}\right) + \left(\frac{1}{3\sqrt{2}}\right)\left(\frac{2}{3}\right) + \left(-\frac{4}{3\sqrt{2}}\right)\left(\frac{1}{3}\right) = 0$$

$$\langle \mathbf{u}_1, \mathbf{u}_3 \rangle = \left(\frac{1}{3\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{3\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) + \left(-\frac{4}{3\sqrt{2}}\right)(0) = 0$$

$$\langle \mathbf{u}_2, \mathbf{u}_3 \rangle = \left(\frac{2}{3}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{2}{3}\right)\left(-\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{3}\right)(0) = 0$$

Chapter 5, Section 6

3. We start by forming $\mathbf{u}_1$:

$$\boxed{\mathbf{u}_1 = \frac{\mathbf{x}_1}{||\mathbf{x}_1||} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}}$$

Now we find $\mathbf{u}_2$:

$$\mathbf{p}_1 = \langle \mathbf{x}_2, \mathbf{u}_1 \rangle \mathbf{u}_1 = \frac{2}{3}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$$\mathbf{x}_2 - \mathbf{p}_1 = \frac{1}{3}\begin{bmatrix} 10 \\ 5 \\ 10 \end{bmatrix}$$

$$\boxed{\mathbf{u}_2 = \frac{\mathbf{x}_2 - \mathbf{p}_1}{||\mathbf{x}_2 - \mathbf{p}_1||} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}}$$

Finally, we find $\mathbf{u}_3$:

$$\mathbf{p}_2 = \langle \mathbf{x}_3, \mathbf{u}_1 \rangle \mathbf{u}_1 + \langle \mathbf{x}_3, \mathbf{u}_2 \rangle \mathbf{u}_2 = \frac{1}{3} \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix}$$

$$\mathbf{x}_3 - \mathbf{p}_2 = \frac{1}{3} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$\boxed{\mathbf{u}_3 = \frac{\mathbf{x}_3 - \mathbf{p}_2}{||\mathbf{x}_3 - \mathbf{p}_2||} = \frac{1}{3} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}}$$

5. (a) We start by forming $\mathbf{u}_1$ using $\mathbf{x}_1$, the first columns of A:

$$\boxed{\mathbf{u}_1 = \frac{\mathbf{x}_1}{||\mathbf{x}_1||} = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}}$$

Now we find $\mathbf{u}_2$:

$$\mathbf{p}_1 = \langle \mathbf{x}_2, \mathbf{u}_1 \rangle \mathbf{u}_1 = \frac{1}{9} \begin{bmatrix} 10 \\ 5 \\ 10 \end{bmatrix}$$

$$\mathbf{x}_2 - \mathbf{p}_1 = \frac{1}{9} \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}$$

$$\boxed{\mathbf{u}_2 = \frac{\mathbf{x}_2 - \mathbf{p}_1}{||\mathbf{x}_2 - \mathbf{p}_1||} = \frac{1}{3\sqrt{2}} \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}}$$

(b) The columns of Q consist of $\mathbf{u}_1$ and $\mathbf{u}_2$ from part (a):

$$\boxed{Q = \begin{bmatrix} 2/3 & -1/(3\sqrt{2}) \\ 1/3 & 4/(3\sqrt{2}) \\ 2/3 & -1/(3\sqrt{2}) \end{bmatrix}}$$

The entries in the R matrix are:

$$r_{11} = \mathbf{x}_1 = 3$$
$$r_{12} = \langle \mathbf{u}_1, \mathbf{x}_2 \rangle = 5/3$$
$$r_{22} = ||\mathbf{x}_2 - \mathbf{p}_1|| = \sqrt{2}/3$$

Therefore, the R matrix is:

$$\boxed{R = \begin{bmatrix} 3 & 5/3 \\ 0 & \sqrt{2}/3 \end{bmatrix}}$$

(c) The solution to the least squares problem is the solution to the system:

$$R\mathbf{x} = Q^T\mathbf{b}$$

Using back substitution or inverting the R matrix, we find that:

$$\boxed{\mathbf{x} = \begin{bmatrix} 9 \\ -3 \end{bmatrix}}$$

Chapter 6, Section 1

1. (a) To find the eigenvalues, we solve the equation $\det(A - \lambda I) = 0$:

$$\begin{aligned}
\det(A - \lambda I) &= 0 \\
\begin{vmatrix} 3-\lambda & 2 \\ 4 & 1-\lambda \end{vmatrix} &= 0 \\
(3-\lambda)(1-\lambda) - (2)(4) &= 0 \\
\lambda^2 - 4\lambda - 5 &= 0 \\
(\lambda - 5)(\lambda + 1) &= 0 \\
&\boxed{\lambda = 5, -1}
\end{aligned}$$

For $\lambda = 5$, we then solve the system $(A - 5I)\mathbf{x} = \mathbf{0}$:

$$\begin{aligned}
(A - 5I)\mathbf{x} &= \mathbf{0} \\
\begin{bmatrix} -2 & 2 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\Rightarrow \quad -2x_1 + 2x_2 &= 0 \\
x_1 &= x_2
\end{aligned}$$

Let $x_2 = \alpha \Rightarrow x_1 = x_2 = \alpha$. The eigenspace for $\lambda = 5$ is then:

$$\left\{ \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} \middle| \; \alpha \in \mathbb{R} \right\} = \left\{ \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} \middle| \; \alpha \in \mathbb{R} \right\} = \boxed{\text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}}$$

For $\lambda = -1$, we then solve the system $(A + I)\mathbf{x} = \mathbf{0}$:

$$\begin{aligned}
(A + I)\mathbf{x} &= \mathbf{0} \\
\begin{bmatrix} 4 & 2 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\Rightarrow \quad 4x_1 + 2x_2 &= 0 \\
x_1 &= -\frac{1}{2}x_2
\end{aligned}$$

Let $x_2 = 2\alpha \Rightarrow x_1 = -\alpha$. The eigenspace for $\lambda = -1$ is then:

$$\left\{ \begin{bmatrix} -\alpha \\ 2\alpha \end{bmatrix} \middle| \; \alpha \in \mathbb{R} \right\} = \left\{ \alpha \begin{bmatrix} -1 \\ 2 \end{bmatrix} \middle| \; \alpha \in \mathbb{R} \right\} = \boxed{\text{Span}\left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}}$$

(e) To find the eigenvalues, we solve the equation $\det(A - \lambda I) = 0$:

$$\begin{aligned}
\det(A - \lambda I) &= 0 \\
\begin{vmatrix} 1-\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} &= 0 \\
(1-\lambda)(3-\lambda) - (1)(-2) &= 0 \\
\lambda^2 - 4\lambda + 5 &= 0 \\
\lambda &= \frac{4 \pm \sqrt{4^2 - 4(1)(5)}}{2} \\
&\boxed{\lambda = 2 \pm i}
\end{aligned}$$

For $\lambda = 2+i$, we then solve the system $(A-(2+i)I)\mathbf{x} = \mathbf{0}$:

$$\begin{aligned}
(A-(2+i)I)\mathbf{x} &= \mathbf{0} \\
\begin{bmatrix} -1-i & 1 \\ -2 & 1-i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\Rightarrow \quad -2x_1 + (1-i)x_2 &= 0 \\
x_1 &= \frac{1}{2}(1-i)x_2
\end{aligned}$$

Let $x_2 = 2\alpha \Rightarrow x_1 = (1-i)\alpha$. The eigenspace for $\lambda = 2+i$ is then:

$$\left\{ \begin{bmatrix} (1-i)\alpha \\ 2\alpha \end{bmatrix} \middle| \ \alpha \in \mathbb{R} \right\} = \left\{ \alpha \begin{bmatrix} 1-i \\ 2 \end{bmatrix} \middle| \ \alpha \in \mathbb{R} \right\} = \boxed{\text{Span}\left\{ \begin{bmatrix} 1-i \\ 2 \end{bmatrix} \right\}}$$

For $\lambda = 2-i$, we simply take the conjugate to get the eigenspace:

$$\boxed{\text{Span}\left\{ \begin{bmatrix} 1+i \\ 2 \end{bmatrix} \right\}}$$

(g) This matrix is upper triangular so we know that the eigenvalues are along the diagonal: $\boxed{\lambda = 1, 2, 1}$.
For $\lambda = 1$, we then solve the system $(A-I)\mathbf{x} = \mathbf{0}$:

$$\begin{aligned}
(A-I)\mathbf{x} &= \mathbf{0} \\
\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\Rightarrow \quad x_2 + x_3 &= 0 \\
x_2 &= -x_3
\end{aligned}$$

Both x_1 and x_3 are free variables. Let $x_1 = \alpha$ and $x_3 = \beta \Rightarrow x_2 = -x_3 = -\beta$. The eigenspace for $\lambda = 1$ is then:

$$\left\{ \begin{bmatrix} \alpha \\ -\beta \\ \beta \end{bmatrix} \middle| \ \alpha, \beta \in \mathbb{R} \right\} = \left\{ \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \middle| \ \alpha, \beta \in \mathbb{R} \right\} = \boxed{\text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}}$$

For $\lambda = 2$, we then solve the system $(A-2I)\mathbf{x} = \mathbf{0}$:

$$\begin{aligned}
(A-2I)\mathbf{x} &= \mathbf{0} \\
\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\Rightarrow \quad x_3 = 0, \ x_1 - x_2 &= 0
\end{aligned}$$

Let $x_2 = \alpha \Rightarrow x_1 = \alpha$. The eigenspace for $\lambda = 2$ is then:

$$\left\{ \begin{bmatrix} \alpha \\ \alpha \\ 0 \end{bmatrix} \middle| \ \alpha \in \mathbb{R} \right\} = \left\{ \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \middle| \ \alpha \in \mathbb{R} \right\} = \boxed{\text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}}$$