Math 310 Homework 12 Solutions

Chapter 6, Section 2

1. (a) The coefficient matrix is:

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$

Its eigenvalues and eigenvectors are:

$$\lambda = 2 : \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\lambda = 3 : \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The eigenvalues are real and distinct. Therefore, the general solution is:

$$\boxed{\mathbf{y}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

(f) The coefficient matrix is:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 6 \\ 0 & 1 & 3 \end{bmatrix}$$

Its eigenvalues and eigenvectors are:

$$\lambda = 1 : \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$\lambda = 0 : \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$
$$\lambda = 5 : \begin{bmatrix} 1 \\ 8 \\ 4 \end{bmatrix}$$

The eigenvalues are real and distinct. Therefore, the general solution is:

$$\boxed{\mathbf{y}(t) = c_1 e^{t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + c_3 e^{5t} \begin{bmatrix} 1 \\ 8 \\ 4 \end{bmatrix}}$$

2. (b) The coefficient matrix is:

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

Its eigenvalues and eigenvectors are:

$$\lambda = 1 + 2i : \begin{bmatrix} -1 \\ i \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$\lambda = 1 - 2i : \begin{bmatrix} -1 \\ -i \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} - i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The eigenvalues are complex with $a = 1$, $b = 2$, $\mathbf{u} = [-1\ 0]^T$, and $\mathbf{v} = [0\ 1]^T$. Therefore, the general solution is:

$$\mathbf{y}(t) = e^t \left\{ c_1 \left(\cos 2t \begin{bmatrix} -1 \\ 0 \end{bmatrix} - \sin 2t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + c_2 \left(\cos 2t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin 2t \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) \right\}$$

Plugging $t = 0$ into the general solution and equating with the initial condition vector $\mathbf{y}(0) = [1\ -2]^T$ we get:

$$\mathbf{y}(0) = c_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

The solution to the above system of equations is: $c_1 = -1$ and $c_2 = -2$. Therefore, the solution is:

$$\boxed{\mathbf{y}(t) = e^t \left\{ - \left(\cos 2t \begin{bmatrix} -1 \\ 0 \end{bmatrix} - \sin 2t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) - 2 \left(\cos 2t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin 2t \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) \right\}}$$

Chapter 6, Section 3

1. (a) The eigenvalues of A are:

$$\lambda = -1 : \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$\lambda = 1 : \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Therefore, the X and D matrices are:

$$X = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

(d) The eigenvalues of A are:

$$\lambda = 2 : \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$\lambda = 1 : \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$
$$\lambda = -1 : \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$$

Therefore, the X and D matrices are:

$$X = \begin{bmatrix} -1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

26. (a) The definition of e^A is:

$$e^A = I + A + \frac{1}{2!}A^2 + \dots$$

We'll start by computing A^2:

$$A^2 = AA = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This means that every term of the form A^k for $k = 2, 3, 4, \ldots$ will also be the zero matrix. So, the exponential of A is simply:

$$\boxed{e^A = I + A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}}$$

(c) The A^k term in the infinite sum will be of the form:

$$A^k = \begin{bmatrix} 1 & 0 & -k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The exponential of A is then:

$$\begin{aligned} e^A &= I + A + \frac{1}{2!}A^2 + \ldots \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \ldots \\ &= \begin{bmatrix} 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots & 0 & -\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots\right) \\ 0 & 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots & 0 \\ 0 & 0 & 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots \end{bmatrix} \end{aligned}$$

To simplify we note that:

$$\begin{aligned} e^x &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots \\ \Rightarrow \quad e^1 &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots \end{aligned}$$

Therefore, we have:

$$\boxed{e^A = \begin{bmatrix} e & 0 & -e \\ 0 & e & 0 \\ 0 & 0 & e \end{bmatrix}}$$

28. (a) The solution is of the form:

$$\mathbf{y}(t) = Xe^{Dt}X^{-1}\mathbf{y}_0$$

The eigenvalues and eigenvectors of A are:

$$\lambda = 1 : \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda = -1 : \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Therefore, the X and D matrices are:

$$X = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

We then have:

$$X^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad e^{Dt} = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$$

The solution is then:

$$\mathbf{y}(t) = Xe^{Dt}X^{-1}\mathbf{y}_0$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boxed{\mathbf{y}(t) = \begin{bmatrix} 2e^t - e^{-t} \\ e^{-t} \end{bmatrix}}$$