Math 310 Homework 5 Solutions

Chapter 3, Section 1

3. For the set of complex numbers C, addition is defined by: $(a+bi)+(c+di) = (a+c)+(b+d)i$; scalar multiplication is defined by: $\alpha(a+bi) = \alpha a + \alpha bi$. To show that C is a vector space, we must show that the 8 axioms are satisfied.

A1. Let $\mathbf{x} = a + bi$ and $\mathbf{y} = c + di$. Then

$$\begin{aligned}\mathbf{x}+\mathbf{y} &= (a+bi)+(c+di)\\ &= (a+c)+(b+d)i\\ &= (c+a)+(d+b)i\\ &= (c+di)+(a+bi)\\ &= \mathbf{y}+\mathbf{x}\end{aligned}$$

A2. Let $\mathbf{x} = a+bi$, $\mathbf{y} = c+di$, and $\mathbf{z} = e+fi$. Then:

$$\begin{aligned}(\mathbf{x}+\mathbf{y})+\mathbf{z} &= ((a+bi)+(c+di))+(e+fi)\\ &= ((a+c)+(b+d)i)+(e+fi)\\ &= (a+c+e)+(b+d+f)i\\ &= (a+bi)+((c+e)+(d+f)i)\\ &= (a+bi)+((c+di)+(e+fi))\\ &= \mathbf{x}+(\mathbf{y}+\mathbf{z})\end{aligned}$$

A3. Let $\mathbf{x} = a+bi$ and $\mathbf{0} = 0+0i$. Then:

$$\mathbf{x}+\mathbf{0} = (a+bi)+(0+0i) = (a+0)+(b+0)i = a+bi = \mathbf{x}$$

A4. Let $\mathbf{x} = a+bi$ and $-\mathbf{x} = -(a+bi) = -a-bi$. Then:

$$\mathbf{x}+(-\mathbf{x}) = (a+bi)+(-a-bi) = (a-a)+(b-b)i = 0+0i = \mathbf{0}$$

A5. Let $\mathbf{x} = a+bi$, $\mathbf{y} = c+di$, and $\alpha \in \mathbb{R}$. Then:

$$\begin{aligned}\alpha(\mathbf{x}+\mathbf{y}) &= \alpha((a+bi)+(c+di))\\ &= \alpha((a+c)+(b+d)i)\\ &= \alpha(a+c)+\alpha(b+d)i\\ &= \alpha a+\alpha c+\alpha bi+\alpha di\\ &= \alpha(a+bi)+\alpha(c+di)\\ &= \alpha\mathbf{x}+\alpha\mathbf{y}\end{aligned}$$

A6. Let $\mathbf{x} = a+bi$ and $\alpha, \beta \in \mathbb{R}$. Then:

$$\begin{aligned}(\alpha+\beta)\mathbf{x} &= (\alpha+\beta)(a+bi)\\ &= (\alpha+\beta)a+(\alpha+\beta)bi\\ &= \alpha a+\beta a+\alpha bi+\beta bi\\ &= \alpha(a+bi)+\beta(a+bi)\\ &= \alpha\mathbf{x}+\alpha\mathbf{y}\end{aligned}$$

A7. Let $\mathbf{x} = a + bi$ and $\alpha, \beta \in \mathbb{R}$. Then:

$$(\alpha\beta)\mathbf{x} = (\alpha\beta)(a + bi) = \alpha\beta a + \alpha\beta bi = \alpha(\beta a + \beta bi) = \alpha(\beta(a + bi)) = \alpha(\beta\mathbf{x})$$

A8. Let $\mathbf{x} = a + bi$. Then:

$$1 \cdot \mathbf{x} = 1 \cdot (a + bi) = 1 \cdot a + 1 \cdot bi = a + bi = \mathbf{x}$$

6. P is the set of all polynomials. Let $p(x) = A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0$ and $q(x) = B_m x^m + B_{m-1}x^{m-1} + \ldots + B_1 x + B_0$ be any two polynomials of arbitrary degree in P and let $\alpha, \beta \in \mathbb{R}$. Then:

A1.

$$\begin{aligned} p(x) + q(x) &= (A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0) + (B_m x^m + B_{m-1}x^{m-1} + \ldots + B_1 x + B_0) \\ &= A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0 + B_m x^m + B_{m-1}x^{m-1} + \ldots + B_1 x + B_0 \\ &= B_m x^m + B_{m-1}x^{m-1} + \ldots + B_1 x + B_0 + A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0 \\ &= (B_m x^m + B_{m-1}x^{m-1} + \ldots + B_1 x + B_0) + (A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0) \\ &= q(x) + p(x) \end{aligned}$$

A4. Let $-p(x) = -(A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0)$. Then:

$$\begin{aligned} p(x) + (-p(x)) &= A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0 + (-(A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0)) \\ &= A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0 - A_n x^n - A_{n-1}x^{n-1} - \ldots - A_1 x - A_0 \\ &= (A_n - A_n)x^n + (A_{n-1} - A_{n-1})x^{n-1} + \ldots + (A_1 - A_1)x + (A_0 - A_0) \\ &= 0 \end{aligned}$$

A5.

$$\begin{aligned} \alpha(p(x) + q(x)) &= \alpha[(A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0) + (B_m x^m + B_{m-1}x^{m-1} + \ldots + B_1 x + B_0)] \\ &= \alpha(A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0 + B_m x^m + B_{m-1}x^{m-1} + \ldots + B_1 x + B_0) \\ &= \alpha A_n x^n + \alpha A_{n-1}x^{n-1} + \ldots + \alpha A_1 x + \alpha A_0 + \alpha B_m x^m + \alpha B_{m-1}x^{m-1} + \ldots + \alpha B_1 x + \alpha B_0 \\ &= \alpha(A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0) + \alpha(B_m x^m + B_{m-1}x^{m-1} + \ldots + B_1 x + B_0) \\ &= \alpha p(x) + \alpha q(x) \end{aligned}$$

A6.

$$\begin{aligned} (\alpha + \beta)p(x) &= (\alpha + \beta)(A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0) \\ &= (\alpha + \beta)A_n x^n + (\alpha + \beta)A_{n-1}x^{n-1} + \ldots + (\alpha + \beta)A_1 x + (\alpha + \beta)A_0 \\ &= \alpha A_n x^n + \beta A_n x^n + \alpha A_{n-1}x^{n-1} + \beta A_{n-1}x^{n-1} + \ldots + \alpha A_1 x + \beta A_1 x + \alpha A_0 + \beta A_0 \\ &= \alpha A_n x^n + \alpha A_{n-1}x^{n-1} + \ldots + \alpha A_1 x + \alpha A_0 + \beta A_n x^n + \beta A_{n-1}x^{n-1} + \ldots + \beta A_1 x + \beta A_0 \\ &= \alpha(A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0) + \beta(A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0) \\ &= \alpha p(x) + \beta p(x) \end{aligned}$$

11. $V = \{(x_1, x_2) | x_1, x_2 \in \mathbb{R}\}$ where

$$\begin{aligned} (x_1, x_2) + (y_1, y_2) &= (x_1 + y_1, x_2 + y_2) \\ \alpha \circ (x_1, x_2) &= (\alpha x_1, x_2) \end{aligned}$$

V satisfies the closure property since both operations above result in an ordered pair. We must now check the axioms. We will see that A6 is violated:

$$\begin{aligned} (\alpha + \beta) \circ \mathbf{x} &= (\alpha + \beta) \circ (x_1, x_2) \\ &= ((\alpha + \beta)x_1, x_2) \\ &= (\alpha x_1 + \beta x_1, x_2) \end{aligned}$$

However,

$$\begin{aligned}\alpha \circ \mathbf{x} + \beta \circ \mathbf{x} &= \alpha \circ (x_1, x_2) + \beta \circ (x_1, x_2) \\ &= (\alpha x_1, x_2) + (\beta x_1, x_2) \\ &= (\alpha x_1 + \beta x_1, 2x_2)\end{aligned}$$

Therefore, $(\alpha + \beta) \circ \mathbf{x} \neq \alpha \circ \mathbf{x} + \beta \circ \mathbf{x}$ so $\boxed{V \text{ is not a vector space}}$.

Chapter 3, Section 2

1. For each of these, we must check the three conditions for a set being a subspace of a vector space V.
   (a) i. $\mathbf{0} \in S$ since the condition $x_1 + x_2 = 0$ is satisfied when $x_1 = x_2 = 0$
   ii. Let $\mathbf{x} = (x_1, x_2) \in S$ and $\alpha \in \mathbb{R}$. Then:
   $$\mathbf{y} = \alpha\mathbf{x} = \alpha(x_1, x_2) = \alpha(x_1, -x_1) = (\alpha x_1, -\alpha x_1) = (y_1, y_2)$$
   Since $y_1 + y_2 = \alpha x_1 + (-\alpha x_1) = 0$, $\alpha\mathbf{x} \in \mathbb{R}$.
   iii. Let $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in S$. Then:
   $$\mathbf{z} = \mathbf{x} + \mathbf{y} = (x_1, x_2) + (y_1, y_2) = (x_1, -x_1) + (y_1, -y_1) = (x_1 + y_1, -x_1 - y_1) = (z_1, z_2)$$
   Since $z_1 + z_2 = (x_1 + y_1) + (-x_1 - y_1) = 0$, $\mathbf{x} + \mathbf{y} \in \mathbb{R}$.
   Therefore, $\boxed{S \text{ is a subspace of } \mathbb{R}^2}$.
   (b) i. $\mathbf{0} \in S$ since the condition $x_1 x_2 = 0$ is satisfied when $x_1 = x_2 = 0$
   ii. Let $\mathbf{x} = (x_1, x_2) \in S$ and $\alpha \in \mathbb{R}$. Then:
   $$\mathbf{y} = \alpha\mathbf{x} = \alpha(x_1, x_2) = (\alpha x_1, \alpha x_2) = (y_1, y_2)$$
   Since $y_1 y_2 = (\alpha x_1)(\alpha x_2) = \alpha^2 x_1 x_2 = \alpha^2(0) = 0$, $\alpha\mathbf{x} \in \mathbb{R}$.
   iii. Let $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in S$. Then:
   $$\mathbf{z} = \mathbf{x} + \mathbf{y} = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2) = (z_1, z_2)$$
   Since $z_1 z_2 = (x_1 + y_1)(x_2 + y_2) = x_1 x_2 + y_1 y_2 + x_1 y_2 + x_2 y_1 = 0 + 0 + x_1 y_2 + x_2 y_1 = x_1 y_2 + x_2 y_1 \neq 0$, in general. Therefore, in general, $\mathbf{x} + \mathbf{y} \notin \mathbb{R}$.
   Therefore, $\boxed{S \text{ is not a subspace of } \mathbb{R}^2}$.

2. (a) i. $\mathbf{0} \notin S$ since the condition $x_1 + x_3 = 1$ is not satisfied when $x_1 = x_2 = x_3 = 0$
   Therefore, $\boxed{S \text{ is not a subspace of } \mathbb{R}^3}$.
   (b) i. $\mathbf{0} \in S$ since the condition $x_1 = x_2 = x_3$ is satisfied when $x_1 = x_2 = x_3 = 0$
   ii. Let $\mathbf{x} = (x_1, x_2, x_3) \in S$ and $\alpha \in \mathbb{R}$. Then:
   $$\mathbf{y} = \alpha\mathbf{x} = \alpha(x_1, x_2, x_3) = \alpha(x_1, x_1, x_1) = (\alpha x_1, \alpha x_1, \alpha x_1) = (y_1, y_2, y_3)$$
   Since $y_1 = y_2 = y_3$, $\alpha\mathbf{x} \in \mathbb{R}$.
   iii. Let $\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3) \in S$. Then:
   $$\mathbf{z} = \mathbf{x} + \mathbf{y} = (x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1, x_1, x_1) + (y_1, y_1, y_1) = (x_1 + y_1, x_1 + y_1, x_1 + y_1) = (z_1, z_2, z_3)$$
   Since $z_1 = z_2 = z_3$, $\mathbf{x} + \mathbf{y} \in \mathbb{R}$.
   Therefore, $\boxed{S \text{ is a subspace of } \mathbb{R}^3}$.

4. (a) The determinant is 1. Therefore, the matrix is invertible and $\mathbf{x} = \mathbf{0}$ is the only solution to $A\mathbf{x} = \mathbf{0}$. The nullspace of A is:

$$\boxed{N(A) = \{\mathbf{0}\}}$$

(b) $[A|\mathbf{0}] = \left[\begin{array}{cccc|c} 1 & 2 & -3 & -1 & 0 \\ -2 & -4 & 6 & 3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & -3 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$

Since columns 2 and 3 do not contain pivots, x_2 and x_3 are free variables. Let $x_2 = \alpha$ and $x_3 = \beta$. The second row of the row reduced matrix tells us that $x_4 = 0$. The first row tells us that:

$$x_1 + 2x_2 - 3x_3 = 0$$
$$x_1 = -2x_2 + 3x_3$$
$$x_1 = -2\alpha + 3\beta$$

Therefore, the solutions to $A\mathbf{x} = \mathbf{0}$ are:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2\alpha + 3\beta \\ \alpha \\ \beta \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

The nullspace of A is:

$$\boxed{N(A) = \left\{ \alpha \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \middle| \alpha, \beta \in \mathbb{R} \right\}}$$

(c) The row reduced echelon form of $[A|\mathbf{0}]$ is:

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Column 3 does not contain a pivot. Therefore, x_3 is a free variable. Let $x_3 = \alpha$. The row reduced matrix tells us that:

$$x_1 - x_3 = 0 \quad \Rightarrow \quad x_1 = x_3 = \alpha$$
$$x_2 - x_3 = 0 \quad \Rightarrow \quad x_2 = x_3 = \alpha$$

Therefore, the solutions to $A\mathbf{x} = \mathbf{0}$ are:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ \alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The nullspace of A is:

$$\boxed{N(A) = \left\{ \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \middle| \alpha \in \mathbb{R} \right\}}$$

(d) The row reduced echelon form of $[A|\mathbf{0}]$ is:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 5 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Columns 2 and 4 do not contain a pivot. Therefore, x_2 and x_4 are free variables. Let $x_2 = \alpha$ and $x_4 = \beta$. The row reduced matrix tells us that:

$$\begin{aligned} x_1 + x_2 + 5x_4 = 0 \quad &\Rightarrow \quad x_1 = -x_2 - 5x_4 = -\alpha - 5\beta \\ x_3 + 3x_4 = 0 \quad &\Rightarrow \quad x_3 = -3x_4 = -3\beta \end{aligned}$$

Therefore, the solutions to $A\mathbf{x} = \mathbf{0}$ are:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\alpha - 5\beta \\ \alpha \\ -3\beta \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -5 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

The nullspace of A is:

$$\boxed{N(A) = \left\{ \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -5 \\ 0 \\ -3 \\ 1 \end{bmatrix} \middle| \, \alpha, \beta \in \mathbb{R} \right\}}$$

5. (a) The set of polynomials in P_4 of even degree **is not** a subspace of P_4 because it is not closed under addition. For example, let $p(x) = x^2 + x + 1$ and $q(x) = -x^2$. Then $p(x) + q(x) = x + 1$ which has odd degree and, thus, is not in the set.

(b) The set of all polynomials of degree 3 **is not** a subspace of P_4 because it is not closed under addition. For example, let $p(x) = x^3 + 1$ and $q(x) = -x^3$. Then $p(x) + q(x) = 1$ which is not a polynomial of degree 3 and, thus, is not in the set.

(c) The set of all polynomials $p(x)$ in P_4 such that $p(0) = 0$ is $S = \{ax^3 + bx^2 + cx | a, b, c \in \mathbb{R}\}$. Let's check the three conditions for a subspace:

i. $p(x) = 0 \in S$ since a, b, and c can all be 0

ii. Let $p(x) = ax^3 + bx^2 + cx \in S$ and $\alpha \in \mathbb{R}$. Then:

$$q(x) = \alpha p(x) = \alpha(ax^3 + bx^2 + cx) = \alpha ax^3 + \alpha bx + \alpha cx$$

$q(x) \in S$ since it has degree less than 4 and $q(0) = 0$.

iii. Let $p(x) = ax^3 + bx^2 + cx, q(x) = dx^3 + ex^2 + fx \in S$. Then:

$$r(x) = p(x) + q(x) = ax^3 + bx^2 + cx + dx^3 + ex^2 + fx = (a + d)x^3 + (b + e)x^2 + (c + f)x$$

$r(x) \in S$ since it has degree less than 4 and $r(0) = 0$.

Therefore, S **is** a subspace of P_4.

10. (a) Let's construct a matrix A whose columns are the vectors in the given set:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

The row reduced echelon form of A is:

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since each row contains a pivot, the system of equations $A\mathbf{x} = \mathbf{b}$ has a unique solution for every vector $\mathbf{b} \in \mathbb{R}^3$. This means that $\mathbf{b}$ can be written as a linear combination of the columns of A:

$$\mathbf{b} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3$$

where $\mathbf{x} = (\alpha_1, \alpha_2, \alpha_3)^T$. Thus, $\{(1,0,0)^T, (0,1,1)^T, (1,0,1)^T\}$ **is** a spanning set for $\mathbb{R}^3$.

(b) The given set of vectors includes the three vectors in part (a) and an additional vector $(1,2,3)^T$. Because a subset of this set is a spanning set for $\mathbb{R}^3$ as shown in part (a), the given set **is** also a spanning set for $\mathbb{R}^3$. That is, we can take any vector $\mathbf{b} \in \mathbb{R}^3$ and write it as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ as follows:

$$\mathbf{b} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \alpha_4\mathbf{v}_4$$

where we will always choose $\alpha_4 = 0$.

(c) As in part (a), we construct a matrix A whose columns are the vectors in the given set:

$$A = \begin{bmatrix} 2 & 3 & 2 \\ 1 & 2 & 2 \\ -2 & -2 & 0 \end{bmatrix}$$

$\det A = 0$. In fact, the row reduced echelon form of A is:

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The last row contains all zeros. In order for the system $A\mathbf{x} = \mathbf{b}$ to be consistent, there must be a 0 in the last row of the last column of the row echelon form of $[A|\mathbf{b}]$. This will not occur for every $\mathbf{b} \in \mathbb{R}$. Therefore, we cannot guarantee that we can select a set of scalars $\{\alpha_1, \alpha_2, \alpha_3\}$ such that $\mathbf{b}$ is a linear combination of the given set of vectors. Thus, the given set **is not** a spanning set for $\mathbb{R}^3$.

14. (a) For every $p(x) \in P_3$, can we select a set of scalars $\{\alpha_1, \alpha_2, \alpha_3\}$ such that $p(x) = \alpha_1(1) + \alpha_2(x^2) + \alpha_3(x^2 - 2)$? Let $p(x) = ax^2 + bx + c$ where a, b, c can take on any value. Then:

$$\begin{aligned} p(x) = ax^2 + bx + c &= \alpha_1(1) + \alpha_2(x^2) + \alpha_3(x^2 - 2) \\ ax^2 + bx + c &= (\alpha_2 + \alpha_3)x^2 + (0)x + (\alpha_1 - 2\alpha_3) \end{aligned}$$

If the above equation holds, we must then have:

$$\begin{aligned} \alpha_2 + \alpha_3 &= a \\ 0 &= b \\ \alpha_1 - 2\alpha_3 &= c \end{aligned}$$

The second equation above says that b must be 0 but we said that b can take on any value. Therefore, the given set **is not** a spanning set for P_3.

(b) Following the argument in part (a), we have:

$$\begin{aligned} p(x) = ax^2 + bx + c &= \alpha_1(2) + \alpha_2(x^2) + \alpha_3(x) + \alpha_4(2x+3) \\ ax^2 + bx + c &= (\alpha_2)x^2 + (\alpha_3 + 2\alpha_4)x + (2\alpha_1 + 3\alpha_4) \end{aligned}$$

If the above equation holds, we must then have:

$$\begin{aligned} \alpha_2 &= a \\ \alpha_3 + 2\alpha_4 &= b \\ 2\alpha_1 + 3\alpha_4 &= c \end{aligned}$$

If we let $\alpha_4 = \alpha$ where $\alpha \in \mathbb{R}$ then we have:

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T = \left(-\frac{3}{2}\alpha, a, b - 2\alpha, \alpha\right)^T$$

Therefore, the given set **is** a spanning set for P_3.

(c) Following the argument in part (a), we have:

$$\begin{aligned} p(x) = ax^2 + bx + c &= \alpha_1(x+2) + \alpha_2(x+1) + \alpha_3(x^2-1) \\ ax^2 + bx + c &= (\alpha_3)x^2 + (\alpha_1 + \alpha_2)x + (2\alpha_1 + \alpha_2 - \alpha_3) \end{aligned}$$

If the above equation holds, we must then have:

$$\begin{aligned} \alpha_3 &= a \\ \alpha_1 + \alpha_2 &= b \\ 2\alpha_1 + \alpha_2 + \alpha_3 &= c \end{aligned}$$

The following is a solution to the above system of equations:

$$(\alpha_1, \alpha_2, \alpha_3)^T = (a - b + c, -a + 2b - c, a)^T$$

Therefore, the given set **is** a spanning set for P_3.