Math 310 Homework 6 Solutions

Chapter 3, Section 3

1. (a) **The vectors are LI** because they are not multiples of each other.

   (c) **The vectors are not LI**. If we consider the system of equations $A\mathbf{x} = \mathbf{0}$ where

$$A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3] = \begin{bmatrix} -2 & 1 & 2 \\ 1 & 3 & 4 \end{bmatrix}$$

   we know that there are (at most) 2 pivots in the row reduced form of A. Therefore, there will be (at least) one free variable in the set of all solutions to $A\mathbf{x} = \mathbf{0}$ which means that there are infinite solutions. Thus, we can find scalars c_1,c_2,c_3 not all zero such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$.

2. (a) Let $A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Since $\det A = 1$, A^{-1} exists and $\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}$ is the only solution to $A\mathbf{x} = \mathbf{0}$. Therefore, **the vectors are LI**.

   (b) Let $A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4] = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \end{bmatrix}$. Since the first three vectors are LI (see part (a)), the row reduced form of A will contain a pivot in every row. Therefore, x_4 will be a free variable in the set of all solutions to $A\mathbf{x} = \mathbf{0}$ which means that there are infinite solutions to $A\mathbf{x} = \mathbf{0}$. Thus, **the vectors are not LI**.

   (c) **The vectors are not LI**.

6. (a) It's easiest to use the Wronskian here:

$$W = \begin{vmatrix} 1 & x^2 & x^2 - 2 \\ 0 & 2x & 2x \\ 0 & 2 & 2 \end{vmatrix} = 0$$

   Since $W = 0$ for all x, **the set of functions are not LI**.

   (b)

$$W = \begin{vmatrix} 2 & x^2 & x & 2x + 3 \\ 0 & 2x & 1 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

   Since $W = 0$ for all x, **the set of functions are not LI**.

7. (a)

$$W = \begin{vmatrix} \cos \pi x & \sin \pi x \\ -\pi \sin \pi x & \pi \cos \pi x \end{vmatrix} = \pi$$

   Since $W \neq 0$ for all x, **the functions are LI**.

   (b)

$$W = \begin{vmatrix} x^{3/2} & x^{5/2} \\ (3/2)x^{1/2}x & (5/2)x^{3/2} \end{vmatrix} = x^3$$

   Since $W \neq 0$ for $x = 1$, **the functions are LI**.

   (c)

$$W = \begin{vmatrix} 1 & e^x + e^{-x} & e^x - e^{-x} \\ 0 & e^x - e^{-x} & e^x + e^{-x} \\ 0 & e^x + e^{-x} & e^x - e^{-x} \end{vmatrix} = -4$$

   Since $W \neq 0$ for all x, **the functions are LI**.

11. Let's say we have a set of n vectors one of which is the zero vector: $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}, \mathbf{0}\}$. Then we can find a set of scalars $c_1, c_2, \ldots, c_{n-1}, c_n$ not all zero such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_{n-1}\mathbf{v}_n + c_n\mathbf{0} = \mathbf{0}$$

by choosing c_n to be anything but 0. Therefore, the set of vectors in S are LD.

Chapter 3, Section 4

3. (a) $\mathbf{x}_1$ and $\mathbf{x}_2$ are LI because they are not multiples of each other. We know by Theorem 3.4.3 that any set of 2 LI vectors spans $\mathbb{R}^2$ since $\dim \mathbb{R}^2 = 2$. Therefore, $\mathbf{x}_1$ and $\mathbf{x}_2$ form a basis for $\mathbb{R}^2$.

   (b) $\mathbf{x}_1$,$\mathbf{x}_2$,$\mathbf{x}_3$ are LD because $\mathbf{x}_3$ can be written as a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$

   (c) The dimension of $V = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is 2 since $\mathbf{x}_1$ and $\mathbf{x}_2$ form a basis for V.

5. (a) $\mathbf{x}_3 = 4\mathbf{x}_1 - 2\mathbf{x}_2 \Rightarrow \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are LD

   (b) $\mathbf{x}_1$ and $\mathbf{x}_2$ are not multiples of each other $\Rightarrow$ they are LI

   (c) The dimension of $V = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is 2 since $\mathbf{x}_1$ and $\mathbf{x}_2$ are LI and form a spanning set for V.

   (d) V is a plane that goes through the origin.

8. (a) Consider $A = [\mathbf{x}_1\ \mathbf{x}_2] = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 4 \end{bmatrix}$. The row reduced echelon form of A is:

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

   Since we have a row of zeros, we cannot write every vector $\mathbf{b} \in \mathbb{R}^3$ as a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$. Therefore, $\mathbf{x}_1$ and $\mathbf{x}_2$ do not span $\mathbb{R}^3$.

   (b) $X = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3]$ would have to be nonsingular in order for $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ to form a basis for $\mathbb{R}^3$.

   (c) Let's choose $\mathbf{x}_3 = [1, 0, 0]^T$. The vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are LI since:

$$\begin{vmatrix} 1 & 3 & 1 \\ 1 & -1 & 0 \\ 1 & 4 & 0 \end{vmatrix} = 5$$

   implies that $\mathbf{x} = 0$ is the only solution to $A\mathbf{x} = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3]\mathbf{x} = \mathbf{0}$.

10. We know that $\dim \mathbb{R}^3 = 3$ so any basis for $\mathbb{R}^3$ must contain 3 LI vectors. Therefore, any 3 LI vectors chosen from the list will form a basis. One such set is $\{\mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5\}$.

14. (a) We are told that the set $\{x, x-1, x^2+1\}$ spans $\mathbb{R}^3$. Now the question is whether or not these are also LI. If so, they will form a basis for $V = \text{Span}\{x, x-1, x^2+1\}$. Let's use the Wronskian:

$$W = \begin{vmatrix} x & x-1 & x^2+1 \\ 1 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2$$

   Since $W \neq 0$ for all x, the functions are LI. Therefore, they form a basis for V.

(b) We are told that the set $\{x, x-1, x^2+1, x^2-1\}$ spans $\mathbb{R}^3$. Now the question is whether or not these are also LI. If so, they will form a basis for $V = \text{Span}\{x, x-1, x^2+1, x^2-1\}$. Let's use the Wronskian:

$$W = \begin{vmatrix} x & x-1 & x^2+1 & x^2-1 \\ 1 & 1 & 2x & 2x \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

Since $W = 0$ for all x, the functions are not LI. Therefore, they cannot form a basis for V. If we get rid of $x^2 - 1$, we get the same set of functions as in part (a) which we know are LI. They also span V. Therefore, they form a basis for V and $\dim V = 3$.