Math 310 Homework 7 Solutions

Chapter 3, Section 5

1. (a) $U = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

2. (a) $U^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

3. (a) The transformation equation is $V\mathbf{c} = U\mathbf{d}$ where

$$V = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}, \; U = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

The transformation matrix from $[\mathbf{v}_1, \mathbf{v}_2]$ to $[\mathbf{u}_1, \mathbf{u}_2]$ is:

$$S = U^{-1}V$$

$$S = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$$

$$S = \begin{bmatrix} \frac{5}{2} & \frac{7}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

4. The transformation matrix from the standard basis to E is:

$$U^{-1} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$$

Then we have:

$$\mathbf{x}_E = U^{-1}\mathbf{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\mathbf{y}_E = U^{-1}\mathbf{y} = \begin{bmatrix} 5 \\ -8 \end{bmatrix}$$

$$\mathbf{z}_E = U^{-1}\mathbf{z} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

Chapter 3, Section 6

2. (a) We construct a matrix A whose columns are the given vectors:

$$A = \begin{bmatrix} 1 & 2 & -3 \\ -2 & -2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

Since $\det A = 24$ we know that $\mathbf{x} = \mathbf{0}$ is the only solution to $A\mathbf{x} = \mathbf{0}$. Therefore, the vectors are LI. They also form a spanning set for the subspace and, thus, form a basis for it. The dimension of the subspace is 3 since there are 3 vectors in the basis.

(b) The same argument is made here as in (a) (with $\det A = -3$).

3. (a) The reduced row echelon form of A is:

$$U = \begin{bmatrix} 1 & 2 & 0 & 5 & -3 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Columns 2, 4, and 5 correspond to the free variables.

$$\begin{aligned} \text{Col } 2 &= 2\text{Col } 1 \\ \text{Col } 4 &= 5\text{Col } 1 - \text{Col } 3 \\ \text{Col } 5 &= -3\text{Col } 1 + 2\text{Col } 3 \end{aligned}$$

(b) Columns 1, 3, and 6 of A correspond to the lead variables of U. The remaining columns of A can be written as linear combinations of Columns 1, 3, and 6 as follows:

$$\begin{aligned} \text{Col } 2 &= 2\text{Col } 1 \\ \text{Col } 4 &= 5\text{Col } 1 - \text{Col } 3 \\ \text{Col } 5 &= -3\text{Col } 1 + 2\text{Col } 3 \end{aligned}$$

which are the same equations we had in (a).

4. (c) Since $\det A = 5$, A^{-1} exists and the solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = A^{-1}\mathbf{b}$. Thus, $\mathbf{b}$ is in the column space of A and $A\mathbf{x} = \mathbf{b}$ is consistent.

(d) $\mathbf{b}$ cannot be written as a linear combination of the columns of A. Therefore, $\mathbf{b}$ is not in the column space of A and the system $A\mathbf{x} = \mathbf{b}$ is not consistent.

5. (a) In (c), there will only be one solution.