Math 310 Homework 8 Solutions

Chapter 4, Section 1

5. (a) Is $L(\mathbf{x}) = [x_2, x_3]^T$ a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^2$? Yes, the two linearity properties are satisfied:

$$L(\mathbf{x}+\mathbf{y}) = L\left(\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix}\right) = \begin{bmatrix} x_2+y_2 \\ x_3+y_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_2 \\ y_3 \end{bmatrix} = L(\mathbf{x}) + L(\mathbf{y})$$

$$L(\alpha\mathbf{x}) = L\left(\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{bmatrix}\right) = \begin{bmatrix} \alpha x_2 \\ \alpha x_3 \end{bmatrix} = \alpha \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \alpha L(\mathbf{x})$$

(c) Is $L(\mathbf{x}) = [1+x_1, x_2]^T$ a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^2$? No, the linearity properties are not satisfied:

$$L(\mathbf{x}+\mathbf{y}) = L\left(\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix}\right) = \begin{bmatrix} 1+x_1+y_1 \\ x_2+y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1+x_1 \\ x_2 \end{bmatrix}}_{=L(\mathbf{x})} + \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\neq L(\mathbf{y})}$$

6. (a) Is $L(\mathbf{x}) = [x_1, x_2, 1]^T$ a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$? No, the linearity properties are not satisfied:

$$L(\mathbf{x}+\mathbf{y}) = L\left(\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix}\right) = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}}_{=L(\mathbf{x})} + \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix}}_{\neq L(\mathbf{y})}$$

(b) This is a linear transformation.

7. (d) $L(A) = 2A$ is a linear operator on $\mathbb{R}^{n\times n}$ because the linearity properties are satisfied:

$$L(A+B) = 2(A+B) = 2A+2B = L(A)+L(B)$$

$$L(\alpha A) = 2(\alpha A) = 2\alpha(A) = \alpha(2A) = \alpha L(A)$$

17. (c) The linear operator on $\mathbb{R}^3$ is $L(\mathbf{x}) = [x_1, x_1, x_1]^T$. The kernel is the set of all solutions to $L(\mathbf{x}) = \mathbf{0}$:

$$L(\mathbf{x}) = \begin{bmatrix} x_1 \\ x_1 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, we must have $x_1 = 0$. However, since the input vectors are in $\mathbb{R}^3$, the values of x_2 and x_3 can be anything. Therefore, the kernel is:

$$\ker(L) = \left\{ \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix} \middle|\, x_2, x_3 \in \mathbb{R} \right\}$$

$$\ker(L) = \left\{ x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \middle|\, x_2, x_3 \in \mathbb{R} \right\}$$

$$\boxed{\ker(L) = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

The range or image of L is:

$$\boxed{\operatorname{im}(L) = \left\{ x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \middle| x_1 \in \mathbb{R} \right\} = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}}$$

Chapter 4, Section 2

2. (c) $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

3. (b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

6. To find the matrix representation of L, we first plug in the standard basis vectors into the mapping:

$$L(\mathbf{e}_1) = L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \mathbf{b}_1 + \mathbf{b}_3 = 1 \cdot \mathbf{b}_1 + 0 \cdot \mathbf{b}_2 + 1 \cdot \mathbf{b}_3$$
$$L(\mathbf{e}_2) = L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \mathbf{b}_2 + \mathbf{b}_3 = 0 \cdot \mathbf{b}_1 + 1 \cdot \mathbf{b}_2 + 1 \cdot \mathbf{b}_3$$

The columns of the A matrix are then:

$$\mathbf{a}_1 = [L(\mathbf{e}_1)]_F = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$
$$\mathbf{a}_2 = [L(\mathbf{e}_2)]_F = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

where $F = [\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3]$. The A matrix is then:

$$\boxed{A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}}$$

18. (a) To find the matrix representation of $L(\mathbf{x}) = [x_3, x_1]^T$, we plug in the $\mathbf{u}$ basis vectors into the mapping:

$$L(\mathbf{u}_1) = L\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1 \cdot \mathbf{b}_1 + 0 \cdot \mathbf{b}_2$$
$$L(\mathbf{u}_2) = L\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -3 \cdot \mathbf{b}_1 + 2 \cdot \mathbf{b}_2$$
$$L(\mathbf{u}_3) = L\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \cdot \mathbf{b}_1 + 0 \cdot \mathbf{b}_2$$

The columns of the A matrix are then:

$$\mathbf{a}_1 = [L(\mathbf{u}_1)]_F = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$
$$\mathbf{a}_2 = [L(\mathbf{u}_2)]_F = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$
$$\mathbf{a}_3 = [L(\mathbf{u}_3)]_F = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

where $F = [\mathbf{b}_1\ \mathbf{b}_2]$. The A matrix is then:

$$\boxed{A = \begin{bmatrix} -1 & -3 & 1 \\ 0 & 2 & 0 \end{bmatrix}}$$

Chapter 4, Section 3

2. The linear transformation is defined by $L(\mathbf{x}) = [-x_1, x_2]^T$.

 (a) The transition matrix from the $\mathbf{v}$ basis to the $\mathbf{u}$ basis is given by $S = V^{-1}U$. The U and V matrices are:

$$U = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},\ V = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

 The inverse of V is:

$$V^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

 Therefore, the transition matrix is:

$$\boxed{S = V^{-1}U = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}}$$

 (b) First, we must find the matrix B that represents L with respect to the $\mathbf{u}$ basis. We start by plugging the $\mathbf{u}$ basis vectors into the mapping:

$$L(\mathbf{u}_1) = L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0 \cdot \mathbf{u}_1 + 1 \cdot \mathbf{u}_2$$
$$L(\mathbf{u}_2) = L\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \cdot \mathbf{u}_1 + 0 \cdot \mathbf{u}_2$$

 The columns of B are:

$$\mathbf{b}_1 = [L(\mathbf{u}_1)]_F = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\mathbf{b}_2 = [L(\mathbf{u}_2)]_F = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

 where $F = [\mathbf{u}_1\ \mathbf{u}_2]$. The matrix B is then:

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

 The matrix A that represents L with respect to the $\mathbf{v}$ basis is then:

$$\boxed{A = SBS^{-1} = SIS^{-1} = SS^{-1} = I}$$

4. The transition matrix from the $\mathbf{v}$ basis to the standard basis is:

$$V = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 1 & 0 & 1 \end{bmatrix}$$

The matrix B representing L with respect to the $\mathbf{v}$ basis is then:

$$\boxed{B = V^{-1}AV = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

5. (a) We start by plugging the basis functions into the mapping and writing the results as linear combinations of the basis functions:

$$L(1) = x\frac{d}{dx}(1) + \frac{d^2}{dx^2}(1) = x(0) + 0 = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$$
$$L(x) = x\frac{d}{dx}(x) + \frac{d^2}{dx^2}(x) = x(1) + 0 = x = 0 \cdot 1 + 1 \cdot x + 0 \cdot x^2$$
$$L(x^2) = x\frac{d}{dx}(x^2) + \frac{d^2}{dx^2}(x^2) = x(2x) + 2 = 2x^2 + 2 = 2 \cdot 1 + 0 \cdot x + 2 \cdot x^2$$

Therefore, the columns of A are:

$$\mathbf{a}_1 = [L(1)]_F = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\mathbf{a}_2 = [L(x)]_F = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$\mathbf{a}_3 = [L(x^2)]_F = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

The matrix A is then:

$$\boxed{A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}$$

(b) Using the basis $[1, x, x^2 + 1]$, we'll get the same $L(p_i(x))$ vectors but we must now write these in terms of the new basis:

$$L(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot (x^2 + 1)$$
$$L(x) = x = 0 \cdot 1 + 1 \cdot x + 0 \cdot (x^2 + 1)$$
$$L(x^2 + 1) = 2x^2 + 2 = 0 \cdot 1 + 0 \cdot x + 2 \cdot (x^2 + 1)$$

Therefore, the columns of B are:

$$\mathbf{b}_1 = [L(1)]_F = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\mathbf{b}_2 = [L(x)]_F = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$\mathbf{b}_3 = [L(x^2 + 1)]_F = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

The matrix B is then:

$$\boxed{B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}$$

(c) The matrix S is the transition matrix from $[1, x, x^2 + 1]$ to $[1, x, x^2]$. Since the latter basis is the standard basis for P_3, the U matrix is the identity matrix: $U = I$. The V matrix is:

$$\boxed{V = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Therefore, the transition matrix S is:

$$\boxed{S = U^{-1}V = I^{-1}V = IV = V}$$

It will work out that $B = S^{-1}AS$.