Math 310 Homework 9 Solutions

Chapter 5, Section 1

1. (a) Since $\mathbf{w} = 3\mathbf{v}$, $\mathbf{v}$ and $\mathbf{w}$ have the same direction and, thus, the angle between them is $0°$.
   (b) Since $\mathbf{v}^T\mathbf{w} = 0$, the angle between the vectors is $90°$.

4. We are given that $||\mathbf{x}|| = 2$ and $||\mathbf{y}|| = 3$. So we know that

$$\mathbf{x}^T\mathbf{y} = ||\mathbf{x}||||\mathbf{y}||\cos\theta = 6\cos\theta$$

Therefore, since $|\cos\theta| \leq 1$ we can say that $\boxed{|\mathbf{x}^T\mathbf{y}| \leq 6}$.

Chapter 5, Section 2

1. (a) $R(A^T) = \text{Span}\left\{\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 8 \end{bmatrix}\right\}$. These vectors form a spanning set but they are linearly dependent (they are multiples of each other). Therefore, to get a basis we will eliminate the second vector. So,

$$\boxed{\text{basis for } R(A^T) \text{ is } \begin{bmatrix} 3 \\ 4 \end{bmatrix}}$$

$N(A) = \text{Span}\left\{\begin{bmatrix} -4 \\ 3 \end{bmatrix}\right\}$. This vector forms a spanning set and is linearly independent. So,

$$\boxed{\text{basis for } N(A) \text{ is } \begin{bmatrix} -4 \\ 3 \end{bmatrix}}$$

$R(A) = \text{Span}\left\{\begin{bmatrix} 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}\right\}$. These vectors form a spanning set but they are linearly dependent (they are multiples of each other). Therefore, to get a basis we will eliminate the second vector. So,

$$\boxed{\text{basis for } R(A) \text{ is } \begin{bmatrix} 3 \\ 6 \end{bmatrix}}$$

$N(A^T) = \text{Span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$. This vector forms a spanning set and is linearly independent. So,

$$\boxed{\text{basis for } N(A^T) \text{ is } \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

2. (a) $S = \text{Span}\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\} = R(A^T)$ where

$$A^T = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad A = [1 \ -1 \ 1]$$

Therefore, since $N(A) = R(A^T)^\perp$ we have $S^\perp = R(A^T)^\perp = N(A)$. A is already in row reduced echelon form. Therefore, x_2 and x_3 are free variables for the system of equations $A\mathbf{x} = \mathbf{0}$. Let $x_2 = \alpha$ and $x_3 = \beta$. We then have $x_1 - x_2 + x_3 = 0$ or $x_1 = x_2 - x_3 = \alpha - \beta$. The nullspace of A and, thus, the orthogonal complement of S is then:

$$S^\perp = N(A) = \left\{ \begin{bmatrix} \alpha - \beta \\ \alpha \\ \beta \end{bmatrix} \middle| \ \alpha, \beta \in \mathbb{R} \right\}$$

$$\boxed{S^\perp = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

3. (b) Using the same argument as used in the previous problem, we find that $S^\perp$ is the nullspace of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$. Therefore, we have:

$$\boxed{S^\perp = \text{Span} \left\{ \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix} \right\}}$$

4. Using the same argument as used in the previous two problems, we find that $S^\perp$ is the nullspace of the matrix $A = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & -2 \end{bmatrix}$. Therefore, we have:

$$\boxed{S^\perp = \text{Span} \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}}$$