Math 310 Quiz 5 Solution

1. TRUE or FALSE. The functions $\sin x$, $\cos x$, and $\sin 2x$ are linearly dependent because

$$\sin 2x = 2 \sin x \cos x$$

Justify your answer.

2. Determine the rank of the following matrix:

$$X = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 3 & 0 \\ 3 & 6 & 9 & 13 & 15 \end{bmatrix}$$

Solution:

1. The statement is FALSE. The simple explanation is that the functions are not linearly dependent because there do not exist c_1, c_2 not both zero such that:

$$\sin 2x = c_1 \sin x + c_2 \cos x$$

The more complicated explanation would be that the Wronskian is not equal to zero for all x:

$$W = \begin{vmatrix} \sin x & \cos x & \sin 2x \\ \cos x & -\sin x & 2\cos 2x \\ -\sin x & -\cos x & -4\sin 2x \end{vmatrix}$$

$$= \sin x (4 \sin 2x \sin x + 2 \cos 2x \cos x) - \cos x(-4 \sin 2x \cos x + 2 \cos 2x \sin x) + \sin 2x(-\cos^2 x - \sin^2 x)$$

$$= 4 \sin 2x \sin^2 x + 2 \cos 2x \sin x \cos x + 4 \sin 2x \cos^2 x - 2 \cos 2x \sin x \cos x - \sin 2x$$

$$= 4 \sin 2x (\sin^2 x + \cos^2 x) - \sin 2x$$

$$= 3 \sin 2x$$

One such x for which W is not zero is $x = \pi/4$.

2. The row reduced echelon form of the matrix is:

$$\text{rref}(X) = \begin{bmatrix} 1 & 2 & 0 & 0 & -25 \\ 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The rank of X, by definition, is the dimension of the row space of X. We also know that the dimension of the row space is the number of non-zero rows in the row reduced echelon form of X. Since there are 3 non-zero rows in $\text{rref}(X)$, we know that the rank of X is 3.