Math 310 Quiz 7 Solution

Let $S = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ be a subspace of $\mathbb{R}^3$.

1. Find a basis for $S^\perp$.
2. What is $\dim S^\perp$?

Solution: We will use the fact that $S = R(A^T)$ where

$$A^T = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Since $S^\perp = R(A^T)^\perp = N(A)$, all we have to do is find a basis for $N(A)$ where $A = [0\ 1\ 1]$. This means solving the system $A\mathbf{x} = \mathbf{0}$:

$$\begin{aligned} A\mathbf{x} &= \mathbf{0} \\ [0\ 1\ 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

Column 2 is the only pivot column. Therefore, x_1 and x_3 are free variables. We let $x_1 = \alpha$ and $x_3 = \beta$. Then from the above equation we get $x_2 = -x_3 = -\beta$. The nullspace of A is then:

$$\begin{aligned} N(A) &= \left\{ \begin{bmatrix} \alpha \\ -\beta \\ \beta \end{bmatrix} \middle| \ \alpha, \beta \in \mathbb{R} \right\} \\ N(A) &= \left\{ \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \middle| \ \alpha, \beta \in \mathbb{R} \right\} \\ N(A) &= \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\} \end{aligned}$$

The vectors that span $N(A)$ and, thus, $S^\perp$ are a spanning set and are linearly independent (since there are only two vectors we know that they are LI since they are not multiples of each other). Therefore, a basis for $S^\perp$ is:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Since there are two vectors in the basis, $\dim S^\perp = 2$.