Math 310 Quiz 8 Solution

1. Use the Gram-Schmidt method to find an orthonormal basis from the basis $\{\mathbf{x}_1, \mathbf{x}_2\}$ where:

$$\mathbf{x}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

2. Find the eigenvalues and eigenvectors of the matrix:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution:

1. Start by computing $\mathbf{u}_1$:

$$\boxed{\mathbf{u}_1 = \frac{\mathbf{x}_1}{||\mathbf{x}_1||} = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}}$$

   Now compute $\mathbf{u}_2$:

$$\mathbf{u}_2 = \frac{\mathbf{x}_2 - \mathbf{p}_1}{||\mathbf{x}_2 - \mathbf{p}_1||}$$
$$\mathbf{p}_1 = \langle \mathbf{x}_2, \mathbf{u}_1 \rangle \mathbf{u}_1$$

   First, compute $\mathbf{p}_1$:

$$\mathbf{p}_1 = \frac{3}{25}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

   Then compute the difference $\mathbf{x}_2 - \mathbf{p}_1$:

$$\mathbf{x}_2 - \mathbf{p}_1 = \frac{1}{25}\begin{bmatrix} 16 \\ -12 \end{bmatrix}$$

   Finally we have:

$$\boxed{\mathbf{u}_2 = \frac{1}{20}\begin{bmatrix} 16 \\ -12 \end{bmatrix} = \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}}$$

2. The matrix is upper triangular so the eigenvalues are on the main diagonal:

$$\boxed{\lambda = 0, \ 1 \text{ (repeated)}}$$

   For $\lambda = 0$ we find an associated eigenvector by solving $(A - 0I)\mathbf{x} = \mathbf{0}$:

$$(A - 0I)\mathbf{x} = \mathbf{0}$$
$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow \ x_1 - x_2 = 0$$
$$0 = 0$$
$$x_3 = 0$$

x_2 is a free variable so we let $x_2 = \alpha$. Then we have $x_1 = x_2 = \alpha$ from the first equation. The last equation tells us $x_3 = 0$. The set of solutions are:

$$\boxed{\mathbf{x} = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}$$

For $\lambda = 1$ we find an associated eigenvector(s) by solving $(A - I)\mathbf{x} = \mathbf{0}$:

$$\begin{aligned} (A - 1I)\mathbf{x} &= \mathbf{0} \\ \begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \Rightarrow \quad -x_2 &= 0 \\ -x_2 &= 0 \\ 0 &= 0 \end{aligned}$$

Both x_1 and x_3 are free variables so we let $x_1 = \alpha$ and $x_3 = \beta$. We also have $x_2 = 0$ from the first (and second) equation. The set of solutions are:

$$\boxed{\mathbf{x} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

Therefore, eigenvectors for A are:

$$\boxed{\underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\lambda=0}, \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\lambda=1}}$$