Math 310 Quiz 9 Solution

Factor the matrix A below into the product XDX^{-1} where D is a diagonal matrix:

$$A = \begin{bmatrix} 2 & -8 \\ 1 & -4 \end{bmatrix}$$

Solution: To factor A we must find its eigenvalues and eigenvectors. The eigenvalues of A are:

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \begin{vmatrix} 2-\lambda & -8 \\ 1 & -4-\lambda \end{vmatrix} &= 0 \\ (2-\lambda)(-4-\lambda) - (-8)(1) &= 0 \\ \lambda^2 + 2\lambda &= 0 \\ \lambda(\lambda+2) &= 0 \\ \lambda = 0,\ \lambda &= 2 \end{aligned}$$

An eigenvector for $\lambda = 0$ is found as follows:

$$\begin{gathered} (A - 0I)\mathbf{x} = \mathbf{0} \\ \begin{bmatrix} 2 & -8 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \Rightarrow \quad x_1 - 4x_2 = 0 \end{gathered}$$

Let $x_2 = \alpha$, then $x_1 = 4x_2 = 4\alpha$. Then an eigenvector for $\lambda = 0$ is $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$.
An eigenvector for $\lambda = -2$ is found as follows:

$$\begin{gathered} (A + 2I)\mathbf{x} = \mathbf{0} \\ \begin{bmatrix} 4 & -8 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \Rightarrow \quad x_1 - 2x_2 = 0 \end{gathered}$$

Let $x_2 = \alpha$, then $x_1 = 2x_2 = 2\alpha$. Then an eigenvector for $\lambda = -2$ is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.
The X and D matrices are:

$$X = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

The inverse of X is:

$$X^{-1} = \frac{1}{\det A} \text{adj } A = \frac{1}{2} \begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix}$$