Math 310: **Hour Exam 1** (Solutions)
Prof. S. Smith: Fri 13 October 2000

Problem 1: (a) Using either Gaussian or Gauss-Jordan elimination, find all solutions of the linear equation system $Ax = b$ determined by the following augmented matrix: $(A|b) = \left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 2 & -1 & 1 & 3 \\ 4 & -3 & 1 & 0 \end{array}\right)$.

SHOW the steps of the method you use.

Use row operations (Gauss-Jordan) $\overset{A_2^{-2\times1}\ A_3^{-4\times1}}{\rightarrow} \left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 1 & 1 & 5 \\ 0 & 1 & 1 & 4 \end{array}\right) \overset{A_1^{1\times2}\ A_3^{-1\times2}}{\rightarrow} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & -1 \end{array}\right)$

The bottom row says $0 = 1$, so there are NO solutions.

(b) Which two row operations will bring the matrix $A = \begin{pmatrix} 1 & -4 & 0 \\ 0 & 1 & 1 \\ 0 & 5 & 5 \end{pmatrix}$ into row-reduced echelon form U ? Give elementary row matrices accomplishing those operations by left multiplication; this, give E_1 and E_2 such that $E_2 E_1 A = U$.

Row operations: $A_1^{4\times2}$ *and* $A_3^{-5\times2}$. *Matrices:* $E_1 = \begin{pmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *and* $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$

Problem 2: (a) Is the matrix $E_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$ row-equivalent to the identity matrix I_3 ?

Say why/why not.

No: for example, $det(A) = 0$, *so* A *is singular; so the row-reduced echelon form of* A *has a row of zeros (or, only 2 pivots), and hence cannot be the identity* I_3.

(b) Given the augmented matrix $(A|b) = \left(\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & 3 & 7 \end{array}\right)$, find the inverse (by any method) of the coefficient matrix A; use A^{-1} to give a solution of the corresponding linear system $Ax = b$.

By adjoint method, $A^{-1} = -\begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}$, *so* $x = A^{-1}b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$,

Problem 3: (a) Find the determinant of the product AB, where

$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 5 & 4 \\ 0 & 6 & 5 \\ 0 & 0 & 1 \end{pmatrix}$.

SHOW the steps you used ("calculator" is not sufficient for credit here).

As A *is triangular,* $det(A)$ *is the product down the diagonal, namely* $1.2.1 = 2$.
Similarly $det(B) = 1.6.1 = 6$. *So* $det(AB) = det(A)det(B) = 2.6 = 12$.

(b) For $A = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}$, find the adjoint $adj(A)$, and then the inverse A^{-1}.

We need the transpose of the matrix of cofactors, namely $adj(A) = \begin{pmatrix} 5 & -4 \\ -3 & 2 \end{pmatrix}$.

Then $\det(A) = 2.5 - 3.4 = -2$, $A^{-1} = \frac{1}{det(A)} adj(A) = -\frac{1}{2} \begin{pmatrix} 5 & -4 \\ -3 & 2 \end{pmatrix}$.

Problem 4: (a) In the space $\mathbf{R}^{2\times 2}$ of all 2x2 matrices, show that the set of all upper-triangular matrices forms a subspace. What is the dimension of this subspace.

(+) Add two general upper-triangular matrices: $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} + \begin{pmatrix} d & e \\ 0 & f \end{pmatrix} = \begin{pmatrix} a+d & b+e \\ 0 & c+f \end{pmatrix}$; *so the sum is also upper-triangular.*

(sc.mult.) Multiply a scalar times a general upper-triangular matrix: $f.\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} fa & fb \\ 0 & fc \end{pmatrix}$; *so the product is also upper-triangular.*

Dimension: the "free variables" are a, b, c *above, so the dimension is* 3.

(b) What is the dimension of the span of the columns of the matrix $A = \begin{pmatrix} 1 & 1 & 3 & 0 \\ 0 & 2 & 4 & 2 \\ 1 & 3 & 7 & 2 \end{pmatrix}$?

(Explain how you know this is the dimension).

The dimension is 2. One way: The row-reduced echelon form of A *is* $\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Only 2 pivots, so dim(col.space) = 2. The first two columns give one basis for the col.space).

Problem 5: (a) Find a basis for the row space of the matrix $A = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 2 & 3 & 1 & -4 \\ 3 & 4 & 3 & -5 \end{pmatrix}$.

(Say why you know your answer gives a basis for the row space).

The row-reduced echelon form of A *is* $\begin{pmatrix} 1 & 0 & 5 & 1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Two pivots—so first two rows of rref give one basis for row space. Or, first two rows of A.

(b) Find the matrix of transition from the "old" basis given by the standard basis of $\mathbf{R}^2$ (namely $(1,0)^T$ and $(0,1)^T$) to the "new" basis given by $(3,5)^T$ and $(1,2)^T$.

One way: The matrix $[new]_{old}$ *is given by* $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$,

so the transition matrix $[old]_{new}$ *from old to new is given by its inverse, namely* $\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$.

What are the coordinates of $(4,6)^T$ in this new basis?

One way: Multiply transition matrix by old coordinates $(4,6)^T$ *to get new coordinates* $(2,-2)^T$.