Math 310: **Hour Exam 1** (Solutions)
Prof. S. Smith: Wed 18 October 2006

Problem 1: (a) Using Gaussian elimination, find the row echelon form of the following augmented matrix: $(A|b) = \left(\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 1 & 2 & 0 & 5 \\ 0 & -1 & 2 & 0 \end{array}\right)$. Show the STEPS you use.

$$\overset{A_2^{-1\times 1}}{\to} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right) \overset{A_3^{1\times 2}}{\to} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

(b) Using your answer in (a), perform back substitution to give all SOLUTIONS (if any) of the linear equation system $Ax = b$ determined by the augmented matrix $(A|b)$.

Notice column 3 has no pivot, so that variable x_3 is free; call it "α".
Row 2 says $x_2 - 2x_3 = 0$, so $x_2 = 2x_3 = 2\alpha$.
Row 1 says $x_1 + x_2 + 2x_3 = 5$, so $x_1 = 5 - 2\alpha - 2\alpha = 5 - 4\alpha$.
So infinite number of solutions $(5 - 4\alpha, 2\alpha, \alpha)^T = (5, 0, 0)^T + \alpha(-4, 2, 1)^T$ for all real α.

Problem 2: (a) Find the LU-decomposition (show STEPS) of the matrix $A = \begin{pmatrix} 1 & 0 & 2 \\ -2 & 4 & -4 \\ 0 & 4 & 6 \end{pmatrix}$.

We apply $A_2^{2\times 1}$ to obtain $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 0 \\ 0 & 4 & 6 \end{pmatrix}$ and then $A_3^{-1\times 2}$ to obtain $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ as U.

The product of the matrices for the inverses, namely $A_2^{-2\times 1}$ and $A_3^{1\times 2}$, gives $L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.

(b) Find the inverse (any method, but show STEPS) of the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

Probably quickest is via chapter 1, row-reduce the "big" augmented matrix $(A|I)$ to $(I|A^{-1})$:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \overset{A_1^{-1\times 2}}{\to} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \overset{A_2^{-1\times 3}}{\to} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right).$$

So $A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$.

Problem 3: (a) Let S be the subSET of $\mathbf{R}^3$ consisting of all 3-vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ satisfying the condition $x_3 = x_1 + x_2$. Show that S is a subSPACE of $\mathbf{R}^3$.

A general vector in S has the form $\begin{pmatrix} a \\ b \\ a+b \end{pmatrix}$.

(closure, +) Take two general vectors in S: $\begin{pmatrix} a \\ b \\ a+b \end{pmatrix}$, $\begin{pmatrix} c \\ d \\ c+d \end{pmatrix}$, *and add;*

their sum is $\begin{pmatrix} a+c \\ b+d \\ (a+b)+(c+d) \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \\ (a+c)+(b+d) \end{pmatrix}$ *which is also in S.*

(closure, sc.mult.) For a general scalar c, and general vector $\begin{pmatrix} a \\ b \\ a+b \end{pmatrix}$ *in S,*

the scalar multiple is $\begin{pmatrix} ca \\ cb \\ c(a+b) \end{pmatrix} = \begin{pmatrix} ca \\ cb \\ ca+cb \end{pmatrix}$, *which is also in S.*

(b) What is the rank of the matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 0 & 2 & 2 \end{pmatrix}$?
Give a basis for the span of the columns of A (column space).

The rank is 2; for example the row-reduced echelon form of A is $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ *with 2 pivots.*
Just need two LI columns. In this example, in fact any pair of columns from A will work.

Problem 4: (a) Are the vectors

$v_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$, and $v_3 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ linearly independent in $\mathbf{R}^3$? (Why/why not?)

NO: (short) Notice $v_3 = v_2 - v_1$.

(longer) Set a linear combination equal to zero, get system: $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & 0 \\ 2 & 4 & 2 & 0 \end{array}\right)$

Row-reduce to $\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$*; no pivot in 3rd column, so infinite solutions, hence lin.dep.*

(b) Inside the space $\mathbf{R}^{2\times 2}$ of 2×2 matrices,

let S denote the subspace consisting of symmetric matrices, which have the general form $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$.

Do $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, and $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ give a spanning set for S? (Why/why not)

YES: Set an unknown linear combination equal to the general vector:

$$c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \text{ so } \begin{pmatrix} c_1 + c_2 + c_3 & c_2 + c_3 \\ c_2 + c_3 & c_3 \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

We obtain an equation from each of the four matrix positions:

$$\begin{aligned} c_1 + c_2 + c_3 &= a \\ c_2 + c_3 &= b \text{ (twice, from } (1,2) \text{ and } (2,1) \text{ positions)} \\ c_3 &= c \end{aligned}$$

And we CAN solve. Just quickly by hand: $c_3 = c$*;* $c_2 = b - c$*; and* $c_1 = a - (b - c) - c = a - b$.

Or by the usual full method: $\left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 1 & c \end{array}\right)$ *has row-reduced echelon form* $\left(\begin{array}{ccc|c} 1 & 0 & 0 & a-b \\ 0 & 1 & 0 & b-c \\ 0 & 0 & 1 & c \end{array}\right)$.

Problem 5: (a) Recall that the space $\mathcal{P}_3$ consists of polynomials of degree less than 3.
Find a basis for the subspace S of $\mathcal{P}_3$ given by polynomials with constant term equal to zero.
What is the dimension of S ? (Say WHY your choice is a basis.)

A general polynomial with zero constant term has form $ax + bx^2$.
So choosing a, b *successively in the "standard basis" way, we get 2 "vectors" in a basis:*
$a = 1, b = 0$ *gives* x*; and* $b = 1, a = 0$ *gives* x^2*. Now, why basis?*
(spanning set:) the lin.comb. with coefficients a, b *gives the general* $ax + bx^2$ *above.*
(linearly independent:) Setting that lin.comb., namely $ax + bx^2$*, equal to* 0 *forces* $a = b = 0$.

(b) Find the change-of-basis matrix (transition matrix)

from old basis given by the standard basis $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to the new basis given by $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

(You need to find coordinates of the old basis in terms of the new basis.)

Directly: just solve $\left(\begin{array}{cc|c} 2 & 3 & 1 \\ 1 & 1 & 0 \end{array}\right)$ *and* $\left(\begin{array}{cc|c} 2 & 3 & 0 \\ 1 & 1 & 1 \end{array}\right)$ *to get the two columns of* $\begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$.

Or: invert the transition matrix in the other ("easy") direction: $\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$.