Math 310 **Hour Exam 1** Instructor: Phil Grizzard Wed 4 October 2006

(Solutions)

1.(*20pts*) In the vector space $V = \mathbb{R}^3$, let $S = \{(x_1, x_2, x_3)^T | x_1 x_2 = x_3\}$. Prove or disprove that the subset S is a subspace of V.

S is nonempty: for example, $(0,0,0)^T \in S$. We can represent arbitrary elements of S by $(a, b, ab)^T$ and $(c, d, cd)^T$. Each of these is in S, but their sum $(a+c, b+d, ab+cd)$ is in S only if $(a+c)(b+d) = ab+cd \iff ad = cb = 0$. This is not true in general. For example, consider $\mathbf{x} = (1,1,1)^T$ and $\mathbf{y} = (2,2,4)^T$. Both of these are in S, but their sum $\mathbf{x}+\mathbf{y} = (3,3,5) \notin S$. Thus S is not closed under addition, so S is NOT a subspace of V.

2.(*10pts*) Let $M = \begin{pmatrix} 1 & 2 & -3 & -1 \\ -2 & -4 & 6 & 3 \end{pmatrix}$.

(a) Find the nullspace $N(M)$, and give your answer in set notation.

The nullspace of M is the set of vectors $\mathbf{x}$ such that $M\mathbf{x} = 0$. Row Reduce $(M|0)$ to get $\left(\begin{array}{cccc|c} 1 & 2 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$. *General solution:* $\{(-2\alpha + 3\beta, \alpha, \beta, 0)^T | \alpha, \beta \in \mathbb{R}\}$.

(b) What is the dimension of $N(M)$ over $\mathbb{R}$?

Since α and β are both free variables over $\mathbb{R}$, $N(M)$ is 2-dimensional over $\mathbb{R}$.

For the remainder of the exam, let $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 1 & 2 & 2 \end{pmatrix}$.

3.(*10pts*) Let $b = \begin{pmatrix} 3 \\ 8 \\ 7 \end{pmatrix}$.

(a) Using Gauss-Jordan, find the reduced row-echelon form of the augmented matrix $(A|b)$.

Use row operations to get $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{array}\right)$

(b) Give the solutions of the corresponding linear system $A\mathbf{x} = b$.

$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$

4.(*15pts*) Find the determinant of A by each of the following methods:

(a) Using cofactor expansion.

Use row 2 to get $\det A = (-1)^{2+2} \cdot 4 \cdot (2-1) + (-1)^{2+3} \cdot 3 \cdot (2-2) = 4$.

(b) Using elimination.

Our first row reduction step in #3(a) was a type III operation, which does not change the value of the determinant. Thus $\det A = \det \begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix} = 1 \cdot 4 \cdot 1 = 4$.

5.(*15pts*) Use Cramer's Rule to solve the following system of linear equations. Show an outline of your steps, but you may use a calculator for computations.

$$\begin{aligned} x + 2y + z &= 11 \\ 4y + 3z &= 4 \\ x + 2y + 2z &= -1 \end{aligned}$$

We have $A\mathbf{x} = c$, *where* $c = \begin{pmatrix} 11 \\ 4 \\ -1 \end{pmatrix}$. *By #4, we know* $\det A = 4$. *Let* A_i *be the matrix obtained by replacing the* i^{th} *column of* A *by* c. *We use the calculator to find* $\det A_1 = 12$, $\det A_2 = 40$, *and* $\det A_3 = -48$. *Then Cramer's rule tells us that* $x = \frac{\det(A_1)}{\det A}$, *etc. Our solution is* $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \\ -12 \end{pmatrix}$.

6.(*15pts*)

(a) Find A^{-1}. *Use the same row operations as in #3a on the augmented matrix* $(A|I) = \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 4 & 3 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right)$ *to get* $\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{3}{4} & \frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$. *Thus* $A^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} & -\frac{3}{4} \\ -1 & 0 & 1 \end{pmatrix}$.

(b) Use your answer for (a) and your calculator to show your answers for #3(b) and #5 are correct.

$A^{-1}b = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} & -\frac{3}{4} \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 8 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$. $A^{-1}c = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} & -\frac{3}{4} \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 11 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \\ -12 \end{pmatrix}$. *Thus our answers are correct.*

7.(*5pts*) What is the nullspace of the matrix A? Explain why you don't need to make a new calculation. *A is an invertible matrix, as we've shown. Thus the reduced row echelon form of A is the identity matrix* $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. *There are no free variables, so we will get a unique solution to* $A\mathbf{x} = 0$: *namely,* $\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. *So* $N(A) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

8.(*10pts*) Give the LU-decomposition of A. Be sure to check your answer and show your check. *For our first row reduction step in #1a, we replaced row 3 with row 3 - row 1 and got an upper triangular matrix. Thus* $l_{31} = 1$. *Hence* $L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. *The upper triangular matrix we got was* $U = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix}$. *We check:* $LU = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 1 & 2 & 2 \end{pmatrix} = A$.