Math 310: **Hour Exam 1** (Solutions)

Prof. S. Smith: 26 Sept 1994

Problem 1: Use row-reduced echelon form (Gauss-Jordan) to find all solutions of

$$\begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 3 & 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}.$$

$$\overset{E_{1,2}}{\to} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & 1 & 1 \\ 3 & 4 & 2 & 4 \end{array}\right) \overset{A_2^{-2\times1}, A_3^{-3\times1}}{\to} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & 1 & -1 & -5 \end{array}\right) \overset{A_1^{-1\times2}, A_3^{-1\times2}}{\to} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 8 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus x_3 is free variable; with $x_3 = 1$ in $Ax = 0$, homogenous solutions are $\alpha(-2\,1\,1)$.
With $x_3 = 0$ in $Ax = b$, particular solution is $(8\ -5\,0)$. So general: $(8 - 2\alpha, -5 + \alpha, \alpha)$.

Problem 2:

(a) Is $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ invertible? If so, use elementary row operations to find A^{-1}.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \overset{A_2^{-1\times1}}{\to} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \overset{A_3^{-1\times2}}{\to} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right)$$

Now A^{-1} is on the right.

(b) Use row operations to find the LU decomposition of $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

$\overset{A_2^{-1\times1}}{\to} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ *is echelon-form so gives U;*

and from inverse of the row operation we get $L = A_2^{1\times1}(I_3) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Problem 3:

(a) Find the determinant of $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & 1 \end{pmatrix}$.

Is A singular? How many solutions does $Ax = 0$ have?

(top row) $\det(A) = 1(3 \cdot 1 - 1 \cdot 1) - 2(1 \cdot 1 - 1 \cdot 3) = 2 + 4 = 6.$

Thus A is non-*singular; and $Ax = 0$ has unique solution $x = 0$.*

(b) Use Cramer's rule (determinants) to solve $\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

First, $\det(A) = 2 \cdot 2 - 3 \cdot 3 = 4 - 9 = -5$.

Next $\det \begin{pmatrix} 2 & 3 \\ 5 & 2 \end{pmatrix} = 4 - 15 = -11$ *and* $\det \begin{pmatrix} 2 & 2 \\ 3 & 5 \end{pmatrix} = 10 - 6 = 4$.

So $x_1 = \frac{11}{5}$ *and* $x_2 = -\frac{4}{5}$.

Problem 4: Show that the set S consisting of 2x2 symmetric matrices ($A^T = A$) forms a subspace of $\mathbf{R}^{2\times 2}$.

If $A, B \in S$ *then* $A^T = A$, $B^T = B$.

Then $(A + B)^T = A^T + B^T = A + B$; *so* $A + B$ *is symmetric, and hence is also in* S.

Similarly if $A \in S$ *and* c *is scalar, then* $(cA)^T = c(A^T) = c(A) = cA$,

so that cA *is symmetric, and hence is also in* S.