Math 310: **Hour Exam 1** (Solutions)
Prof. S. Smith: Fri 3 March 2000

Problem 1: (a) Using Gauss-Jordan, find the row-reduced echelon form of the following augmented matrix: $(A|b) = \left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 2 & -1 & 1 & 3 \\ 4 & -3 & 1 & 1 \end{array}\right)$.

Use row operations $\overset{A_2^{-2\times 1}\ A_3^{-4\times 1}}{\rightarrow} \left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 1 & 1 & 5 \\ 0 & 1 & 1 & 5 \end{array}\right) \overset{A_1^{1\times 2}\ A_3^{-1\times 2}}{\rightarrow} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right)$

(b) Then give the solutions of the corresponding linear system $Ax = b$.
So solutions are $(4-r, 5-r, r)$ *for free variable* $x_3 = r$.

Problem 2: (a) Give the LU-decomposition of the matrix $A = \begin{pmatrix} 1 & 1 \\ 3 & 7 \end{pmatrix}$.

Use $A_2^{-3\times 1}$ *to get* $U = \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix}$; *and then the inverse of that operation gives* $L = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$

(b) Find the inverse (by any method) of the matrix $A = \begin{pmatrix} 11 & 2 \\ 6 & 1 \end{pmatrix}$,

Quickly, by classical-adjoint method: $A^{-1} = -\begin{pmatrix} 1 & -2 \\ -6 & 11 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 6 & -11 \end{pmatrix}$,

Problem 3: (a) Find the determinant of: $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 5 & 2 \\ 6 & 4 & 0 \end{pmatrix}$.

Down 2nd column: $(-2)(1.4 - 6.2) = 16$.

(b) Use Cramer's rule to solve $\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & 5 & 2 \end{array}\right)$.

$\det(A) = 1.5 - 1.2 = 3$, *so* $x_1 = \frac{1}{3}(4.5 - 2.2) = \frac{16}{3}$ *and* $x_2 = \frac{1}{3}(1.2 - 1.4) = -\frac{2}{3}$.

Problem 4: (a) In the space $\mathcal{P}_4$ of polynomials of degree at most 3, let S be the subSET of polynomials of degree at most 2 (that is, of form $a + bx + cx^2$ for a, b, c real numbers). Show that S is a subSPACE of $\mathcal{P}_4$.

(+)Add two general polynomials in S: $(a+bx+cx^2)+(d+ex+fx^2) = (a+d)+(b+e)x+(c+f)x^2$; *the result still has degree* ≤ 2, *so is also in* S.
(sc.mult.) Multiply general polynomial in S *by scalar* d: $d(a+bx+cx^2) = da + (db)x + (dc)x^2$; *the result still has degree* ≤ 2, *so is also in* S.

(b) Are the columns of $A = \begin{pmatrix} 1 & 1 & 3 & 0 \\ 2 & 1 & 5 & 1 \\ 1 & 1 & 3 & 0 \end{pmatrix}$ a spanning set for $\mathbf{R}^3$? (Show why/why not).

No. One way: row-reduce to $\begin{pmatrix} 1 & 1 & 3 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Only 2 pivots, so dim(col.space) = 2, which is less than the dimension 3 of $\mathbf{R}^3$.

Problem 5: (a) Find a basis for the nullspace of the matrix $A = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 3 & 5 \end{pmatrix}$.

General solution: $(-2a+b, -3a-5b, a, b)^T$.

Choosing $a=1, b=0$ *gives* $(-2,-3,1,0)^T$, *choosing* $a=0, b=1$ *gives* $(1,-5,0,1)^T$.

(Now CHECK this is a basis...this works out easily...)

(b) Find the matrix of transition from the "old" basis given by the standard basis of $\mathbf{R}^2$ (namely $(1,0)^T$ and $(0,1)^T$) to the "new" basis given by $(1,1)^T$ and $(1,-1)^T$.

One way: The matrix $[new]_{old}$ *is given by* $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$,

so the transition matrix $[old]_{new}$ *from old to new is given by its inverse, namely* $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$.

What are the coordinates of $(4,6)^T$ in this new basis?

One way: Multiply transition matrix by old coordinates $(4,6)^T$ *to get new coordinates* $(5,-1)^T$.