Math 310 **Hour Exam 2** Instructor: Phil Grizzard Fri 3 November 2006

(Solutions)

1.(*20pts*) As commonly done in class, we represent an arbitrary vector in $\mathbb{R}^2$ by $\mathbf{x} = (x_1, x_2)^T$. Prove whether or not each of the following is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$.

(a) $L(\mathbf{x}) = (x_1, 0, 0)^T$.

$L(\alpha\mathbf{x} + \mathbf{y}) = L((\alpha x_1 + y_1, \alpha x_2 + y_2)^T) = (\alpha x_1 + y_1, 0, 0)^T = (\alpha x_1, 0, 0)^T + (y_1, 0, 0)^T = \alpha(x_1, 0, 0)^T + (y_1, 0, 0)^T = \alpha L(\mathbf{x}) + L(\mathbf{y})$. Thus L is linear.

(b) $L(\mathbf{x}) = (x_1, 0, 1)^T$.

L is not linear. For example, if $\alpha \neq 1$, then $L(\alpha\mathbf{x}) = (\alpha x_1, 0, 1) \neq \alpha L(\mathbf{x})$. L also fails summation.

2.(*15pts*) Find the distance from the point $p = (3, 3, 3)$ to the plane $x - y + 3z = 0$.

The plane goes through the origin, and its normal vector (obtained from the coefficients of the equation) is $\mathbf{N} = (1, -1, 3)^T$. We let $\mathbf{v}$ be the vector from the origin to our point p, so $\mathbf{v} = (3, 3, 3)^T$. The distance d from $\mathbf{v}$ to the plane is the absolute value of the scalar projection of $\mathbf{v}$ onto $\mathbf{N}$. So $d = \frac{|\mathbf{v}^T\mathbf{N}|}{||\mathbf{N}||} = \frac{9}{\sqrt{11}}$.

3.(*10pts*) Using the basis $\{1, 1 + x, 1 + x^2\}$ for the space of polynomials of degree at most 2, give the coordinates of the "vector" $1 + x + x^2$.

We are trying to find the coordinates $(a, b, c)^T$ such that $1+x+x^2 = a\cdot 1+b\cdot(1+x)+c\cdot(1+x^2)$. We can see immediately that we need $c = 1$ and $b = 1$, and so we have
$1 + x + x^2 = -1 \cdot 1 + 1 \cdot (1 + x) + 1 \cdot (1 + x^2)$.
Thus the coordinates of this vector are $(-1, 1, 1)^T$ in the given basis.

4.(*20pts*) In polynomial space, are the "vectors" $1 + x$, $1 - x$, and $2 + x^2$ linearly independent? Write your solution clearly and precisely.

By definition of linear independence, these vectors are linearly independent $\Leftrightarrow$ given the equation $c_1(1 + x) + c_2(1 - x) + c_3(2 + x^2) = 0$, the only solution is $c_1 = c_2 = c_3 = 0$. The given equation distributes out to $c_1 + c_1x + c_2 - c_2x + 2c_3 - c_3x^2 = 0$. Collecting like terms tells us that $c_1 + c_2 + 2c_3 = 0$, $c_1 - c_2 = 0$, and $c = 0$. Substituting $c = 0$ into $c_1 + c_2 + 2c_3 = 0$ gives $c_1 + c_2 = 0$. Combining this with $c_1 - c_2 = 0$ shows that $c_1 = c_2 = c_3 = 0$. Thus these vectors are linearly independent.

5.(*15pts*)

(a) Find the dimension of the subspace V of $\mathbb{R}^3$, where V is spanned by the vectors:

$$\mathbf{u}_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}.$$

We make matrix A out of these vectors, so $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 4 \\ 1 & 3 & 5 \end{pmatrix}$. Taking the determinant of this gives $\det A = -2 \neq 0$. Hence these vectors are linearly independent, and since there are 3 of them, they form a basis for the entire space $\mathbb{R}^3$, and so $V = \mathbb{R}^3$. Thus the dimension of V is 3.
Alternatively, we could row reduce A, which gives the identity matrix. V is the same as the column space of this matrix, which has dimension 3.

(b) Find the dimension for $V^\perp$, the orthogonal complement of the space V from part (a).

Since we are working in $\mathbb{R}^3$, *the dimension of the entire space is* $n = 3$. *So for any subspace* V *of* $\mathbb{R}^3$, $\dim V + \dim V^\perp = 3$. *We just saw above that* $\dim V = 3$, *and so* $\dim V^\perp = 0$.
We can also see this from the definition: the orthogonal complement of V *is defined to be* $\{\mathbf{x} \in \mathbb{R}^3 | \mathbf{x}^T\mathbf{y} = 0 \text{ for every } \mathbf{y} \in V\}$. *As* V *is actually all of* $\mathbb{R}^3$, *the only vector that satisfies this is the* $\mathbf{0}$ *vector* $\{0,0,0\}^T$, *which has dimension 0.*

6.(*20pts*)
(a) Using the standard basis for $\mathbb{R}^2$, give the matrix A representing the linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $L(\mathbf{x}) = \begin{pmatrix} x_1 - 3x_2 \\ 2x_1 + 5x_2 \end{pmatrix}$.

Apply L *to the standard basis and put into columns to get* $A = \begin{pmatrix} 1 & -3 \\ 2 & 5 \end{pmatrix}$.
It's always good to check: $\begin{pmatrix} 1 & -3 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - 3x_2 \\ 2x_1 + 5x_2 \end{pmatrix} = L(\mathbf{x})$. *Yep.*

(b) Now give the matrix B for the same L as in part (a), but using the basis $(1,1)^T$ and $(1,2)^T$.

First, we make the "new to old" change-of-basis matrix S. *Since* $(1,1)^T = 1 \cdot \mathbf{e}_1 + 1 \cdot \mathbf{e}_2$ *and* $(1,2)^T = 1 \cdot \mathbf{e}_1 + 2 \cdot \mathbf{e}_2$, *we have* $S = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. *Now we just multiply* $B = S^{-1}AS = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -11 & -22 \\ 9 & 17 \end{pmatrix}$.

Optional bonus.(*2pts Extra Credit*)
EC. Name the 2006 World Series Most Valuable Player.
Shortstop and leadoff hitter ***David Eckstein*** *earned MVP honors after sparking the St. Louis Cardinals to the World Championship with 8 hits, 3 runs scored, and 4 runs batted in during the Series. At 5'7", he becomes the smallest player ever to win the award.*
(Check out `http://www2.math.uic.edu/~grizzard/Eckstein.jpg` *for my personal photo of him at Sunday's parade.)*

Your TEST REFLECTION ASSIGNMENT is due on Monday!
It's the exact same assignment as last time. To see the details, go to
`http://www2.math.uic.edu/~grizzard/Teaching/Exams/TestReflection.pdf`
and wherever you see the word "Friday," insert "Monday."