Math 310: **Hour Exam 2** (Solutions)

Prof. S. Smith: Fri 4 Nov 1994

You must **SHOW WORK** to receive credit.

Problem 1:

(a) Are the vectors $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$ linearly independent?

YES: Maybe easiest is $\det(A) = 4 \neq 0$. *Or check rref(A)= identity .*

(b) Do the vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$ form a spanning set for $\mathbf{R}^3$?

NO: find $\det(A) = 0$; *or compute rref(A) as* $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$*—no pivot in 3rd column.*

Problem 2:

(a) Find a basis for the row space of $\begin{pmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{pmatrix}$. Is $(1, 2, 3, 0)$ in this row space?

Compute rref(A) as $\begin{pmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$ *; first two rows give a basis.*

And $(1, 2, 3, 0)$ *is NOT in the space—because of pivots in 1st and 3rd columns, it would have to equal the combination* $1(1, 3, 0, 1) + 3(0, 0, 1, \frac{1}{3})$*; but doesn't.*

(b) Find a basis for the subspace of all symmetric ($A = A^T$) 2x2 matrices. What is the dimension of this subspace?

By the symmetric requirement, these matrices have form $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$.

Choosing $a = 1, b = 0, d = 0$ *etc, get matrices*

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ *for basis. And dimension is 3.*

Problem 3: Find the matrix, in the standard bases, for the linear transformation

$$L\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 2z \\ 3x + y \\ 2x - z \end{pmatrix}.$$

Apply L *to standard basis, put into columns to get* $\begin{pmatrix} 0 & 0 & 2 \\ 3 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$.

Problem 4:

(a) Find an orthonormal basis for the subspace W of $\mathbf{R}^3$ spanned by the row vectors $(1,1,1)$ and $(1,1,0)$.

Let's use Gram-Schmidt, with normalization afterwards. *Call initial vectors* x_1,x_2.

Form $y_2 = x_2 - \frac{x_2 \cdot x_1}{x_1 \cdot x_1} x_1 = (1,1,0) - \frac{2}{3}(1,1,1) = \frac{1}{3}(1,1,-2)$.

Now normalize to get $u_1 = \frac{1}{\sqrt{3}}(1,1,1)$ *and* $u_2 = \frac{1}{\sqrt{6}}(1,1,-2)$.

Note: using $(1,1,0)$ *first leads to nicer basis* $\frac{1}{\sqrt{2}}(1,1,0)$ *and* $(0,0,1)$.

(b) Now find a basis for the subspace $W^\perp$ orthogonal to W in part (a).

Using the vectors in (a) as rows of A and row-reducing gives $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Nullspace has form $a(1,-1,0)$; *normalizing gives basis* $\frac{1}{\sqrt{2}}(1,-1,0)$.

Problem 5: Find the least-squares solution of the inconsistent system

$$\begin{pmatrix} 1 & 0 \\ 1 & 3 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}.$$

What is the error with this approximation?

Multiply A^T by the augmented matrix $[Ab]$ to get

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 3 & 4 \\ 1 & 6 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 9 & 10 \\ 9 & 45 & 42 \end{pmatrix}.$$

Row operation $A_2^{-3\times 1}$ *leads to* $\begin{pmatrix} 3 & 9 & 10 \\ 0 & 18 & 12 \end{pmatrix}$. *This gives* $y = \frac{2}{3}$ *and then* $x = \frac{4}{3}$.

Multiply by A to get approximation $\frac{1}{3}\begin{pmatrix} 4 \\ 10 \\ 16 \end{pmatrix}$, *so error is* $\frac{1}{3}\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$, *of length* $\sqrt{\frac{2}{3}}$.