Math 310: **Final Exam** (Solutions) Prof. S. Smith: Wednesday 13 December 2006

Problem 1: Let $A = \begin{pmatrix} 6 & -4 \\ 3 & -1 \end{pmatrix}$. Work **by hand**; do *not* use a calculator on this problem. (Except to *check* your work.)

(a) Find the characteristic polynomial, and the eigenvalues, of A.

$\det(A - xI) = (6-x)(-1-x) - (-4.3) = (x^2 - 5x - 6) - (-12) = x^2 - 5x + 6 = (x-3)(x-2)$, *so eigenvalues are* $2, 3$.

(b) Find the eigenspaces for those eigenvalues.

For 2: $A - 2.I = \left(\begin{array}{cc|c} 4 & -4 & 0 \\ 3 & -3 & 0 \end{array}\right)$ *has rref* $\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$, *get solutions* $\alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For 3: $A - 31.I = \left(\begin{array}{cc|c} 3 & -4 & 0 \\ 3 & -4 & 0 \end{array}\right)$, *has rref* $\left(\begin{array}{cc|c} 1 & -\frac{4}{3} & 0 \\ 0 & 0 & 0 \end{array}\right)$, *get solutions* $\beta \begin{pmatrix} \frac{4}{3} \\ 1 \end{pmatrix}$.

Problem 2: Given the differential equation system (functions of t): $\begin{pmatrix} y_1' & = & y_1 & -2y_2 \\ y_2' & = & 3y_1 & -4y_2 \end{pmatrix}$.

I GIVE you the information that eigenvalues of the coefficient matrix A for this system are $-1, -2$.

(a) Find eigenvectors for these eigenvalues of A; then use them to give the *general* solution of the system (with undetermined constants c_1, c_2).

For -1, *get* $\alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; *for* -2, *get* $\beta \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix}$, *so can use* $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Then soln.vec. $c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t}$, *so* $y_1 = c_1 e^{-t} + 2c_2 e^{-2t}$ *and* $y_2 = c_1 e^{-t} + 3c_2 e^{-2t}$.

(b) Now find the particular solution (values of c_1, c_2) given initial values $y_1(0) = 2$, $y_2(0) = 3$.

Solve $\left(\begin{array}{cc|c} 1 & 2 & 2 \\ 1 & 3 & 3 \end{array}\right)$ *to get* $c_1 = 0, c_2 = 1$. *So* $y_1 = 2e^{-2t}$ *and* $y_2 = 3e^{-2t}$.

Problem 3:

(a) GIVEN: the eigenvalues of $A = \begin{pmatrix} 2 & -8 \\ 1 & -4 \end{pmatrix}$ are $0, -2$. Diagonalize A: that is, give matrices X, X^{-1}, and D such that $X^{-1}AX = D$ with D a diagonal matrix.

Find eigenvectors for 0, *say* $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$; *and for* -2, *say* $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

We can use $X = \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}$, $X^{-1} = \begin{pmatrix} \frac{1}{2} & -1 \\ -\frac{1}{2} & 2 \end{pmatrix}$ *with* $D = X^{-1}AX = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$.

(b) Let $A = \begin{pmatrix} 4 & 6 & -2 \\ -1 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$. GIVEN: the eigenvalues of A are $2, 2, 1$.

Find the DIMENSIONS of the eigenspaces for these eigenvalues.

Is A diagonalizable? Say why/why not.

Check that $rref(A - 2.I_3)$ *has 2 free variables, so the dimension of the 2-eigenspace is 2. However also* $rref(A - 1.I_3)$ *has 1 free variable, so the dimension of the 1-eigenspace is 1. Then* A *is diagonalizable—since for each eigenvalue, the dimension of the eigenspace is equal to the number of times the eigenvalue appears as a root of the characteristic polynomial. (That is, geometric multiplicity = algebraic multiplicity for each eigenvalue).*

Problem 4: For the symmetric matrix $A = \begin{pmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{pmatrix}$,

I GIVE you the eigenvalues $-2, 7, 7$ of A; and an eigenvector $\begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$ for eigenvalue -2.

(a) Find a basis of the eigenspace of A for eigenvalue 7.

The row-reduced echelon form of $A - (0).I = A$ has $(1, \frac{1}{2}, -1)$ as its only nonzero row.

So eigenvectors are $\begin{pmatrix} -\frac{1}{2}b + c \\ b \\ c \end{pmatrix}$; *and one possible basis is* $\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ *and* $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

(b) Now find an *orthonormal* basis for the 7-eigenspace in (a).
Use it to give an orthogonal diagonalization of A;
that is, find an *orthogonal* matrix X (satisfying $X^{-1} = X^T$) with $X^{-1}AX$ diagonal.
Show WORK (Gram-Schmidt) in obtaining your orthonormal basis (**no** calculators!)

For -2: eigenspace is 1-dimensional; divide original $\begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$ *by its length 3:* $x_3 = \frac{1}{3}\begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$.

For 7: Start with above basis like $v_1 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ *and* $v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Apply Gram-Schmidt: first $q_1 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$; *and then* $q_2 = v_2 - \frac{v_2 \cdot q_1}{q_1 \cdot q_1} q_1$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}}{\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{-1}{5}\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ \frac{2}{5} \\ 1 \end{pmatrix}, \text{ so use } = \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}.$$

Now divide each by its length, to get $x_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ *and* $x_2 = \frac{1}{\sqrt{45}}\begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}$.

So now putting x_3 first, can use $X = \begin{pmatrix} -\frac{2}{3} & -\frac{1}{\sqrt{5}} & \frac{4}{\sqrt{45}} \\ -\frac{1}{3} & \frac{2}{\sqrt{5}} & \frac{2}{\sqrt{45}} \\ \frac{2}{3} & 0 & \frac{5}{\sqrt{45}} \end{pmatrix}$.

Problem 5: (a) Is the matrix $\begin{pmatrix} 3 & 4 \\ 4 & 1 \end{pmatrix}$ positive definite? (Why/why not—any method)

No; it has a negative eigenvalue $-2.1231\ldots$;

Or, the principal minor determinants are 3 *and* $3 \cdot 1 - 4 \cdot 4 = -13$, *and the latter is negative.*

(b) Give the LU-decomposition of the symmetric matrix $A = \begin{pmatrix} 4 & 2 \\ 2 & 10 \end{pmatrix}$.

Then give the Cholesky decomposition of A—that is, find a matrix B such that $A = BB^T$.

The row operation $A_2^{-\frac{1}{2}\times 1}$ *converts* A *to the row echelon form* $\begin{pmatrix} 4 & 2 \\ 0 & 9 \end{pmatrix}$, *which we use for* U.

And then L *is the inverse of the matrix for that operation, namely* $\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix}$.

That is, the LU*-decomposition is given by* $A = LU$: $\begin{pmatrix} 4 & 2 \\ 2 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 0 & 9 \end{pmatrix}$.

Next, we factor $U = DL^T$ $\begin{pmatrix} 4 & 2 \\ 0 & 9 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$

as the product of the diagonal matrix D *of its eigenvalues, with* L^T.

This gives the decomposition $A = LDL^T$: $\begin{pmatrix} 4 & 2 \\ 2 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$.

We then take the "square root" of D: *namely* $E = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$;

and finally we set $B = LE = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$.

We then have the Cholesky decomposition $A = BB^T$: $\begin{pmatrix} 4 & 2 \\ 2 & 10 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$.