Math 310: **Final Exam** (Solutions)

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Problem 1:

(a) Find the eigenvalues of $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 5 & 0 & 4 \end{pmatrix}$.

Using second row, we see

$\det(A - xI) = -(3-x)[(1-x)(4-x) - 5 \cdot 2] = (x-3)(x^2 - 5x - 6) = (x-3)(x+1)(x-6)$

So eigenvalues $-1, 3, 6$.

(b) For each eigenvalue, find a basis for the corresponding eigenspace of A.

Solving $(A - \lambda I)x = 0$ *by Chapter 1 for each* λ *gives the columns of:*

$$\begin{pmatrix} -1 & 0 & \frac{2}{5} \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

(c) Is A diagonalizable? If possible, give X such that $X^{-1}AX = D$ is diagonal.

Yes. The eigenvectors form a basis, so can use matrix in (b) for X.

Problem 2: I *give* you that $X^{-1}AX = D$ where

$$A = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}, X = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, D = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}.$$

(a) Give the general solution of the differential equation system $\begin{array}{rcrcr} y_1' & = & -y_1 & + & 2y_2 \\ y_2' & = & 2y_1 & - & y_2 \end{array}$.

Using eigenvalues $-3, 1$ *and eigenvectors from given diagonalization,*

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t$$

so $y_1 = -c_1 e^{-3t} + c_2 e^t$ *and* $y_2 = c_1 e^{-3t} + c_2 e^t$.

(b) Give the particular solution when $y_1(0) = 3$ and $y_2(0) = 1$.

Putting in $t = 0$ *gives system with augmented matrix* $\begin{pmatrix} -1 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}$,

Then Chapter 1 methods give solution $c_1 = -1$, $c_2 = 2$;

so $y_1 = e^{-3t} + 2e^t$, $y_2 = -e^{-3t} + 2e^t$.

Problem 3:

(a) Let M be the Markov matrix $\begin{pmatrix} .9 & .3 \\ .1 & .7 \end{pmatrix}$.

Find the "steady state" eigenvector for M. (Components of vector should add to 1).

For eigenvalue 1, $M - 1I = \frac{1}{10} \begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix}$,

so eigenvector is span of $(3, 1)^T$. *Hence* $(\frac{3}{4}, \frac{1}{4})$ *is steady-state vector.*

(b) Find eigenvectors and eigenvalues for $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$.
Is A diagonalizable (why/why not) ? If possible diagonalize A.

$\det(A) = (1-x)(-1-x)+1 = x^2-1+1 = x^2$, so eigenvalues are $0, 0$.
Only solutions of $A - 0x = 0$ are multiples of $(1,1)^T$.
Not diagonalizable: (geometric multiplicity = 1) < (algebraic multiplicity = 2).

Problem 4: Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. I *give* you that the eigenvalues are $-1, -1, 2$.

(a) Find eigenvectors for the eigenvalue -1. (I *give* you that a 2-eigenvector is $(1,1,1)^T$).

$A - (-1)I = A + I$ row-reduces to $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,
so a basis for 0-eigenvectors is $(-1,1,0)^T$ and $(0,-1,1)^T$.

(b) Note A is *symmetric*. So find *orthogonal* S with $S^{-1}AS$ diagonal.
(Remember this means the columns of S must be *orthonormal*).

We need to apply Gram-Schmidt to each eigenspace.
For 2, just divide by length to get $\frac{1}{\sqrt{3}}(1,1,1)^T$.
For -1, divide first by length to get $\frac{1}{\sqrt{2}}(-1,1,0)^T$ and then compute
$(0,-1,1)^T - [\frac{1}{\sqrt{2}}(-1,1,0)(0,-1,1)^T]\frac{1}{\sqrt{2}}(-1,1,0)^T = (0,-1,1)^T + \frac{1}{2}(-1,1,0)^T$
$= \frac{1}{2}(-1,-1,2)^T$*; divide by length to get* $\frac{1}{\sqrt{6}}(-1,-1,2)^T$.

So use $S = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}$.

Problem 5:

(a) Find the matrix of the quadratic form $6x^2 + 5y^2 + 6z^2 + 8xy - 4xz + 6yz$, and use determinants to decide if it is/is not positive definite.

$A = \begin{pmatrix} 6 & 4 & -2 \\ 4 & 5 & 3 \\ -2 & 3 & 6 \end{pmatrix}$

Determinants of principal minors are 6, $6.5 - 4.4 = 14$ and $\det(A) = -38$.
The last is negative, so not *positive definite.*

(b) Write $\begin{pmatrix} 4 & 2 \\ 2 & 10 \end{pmatrix}$ as LDL^T for L lower triangular and D diagonal.

Row operation $A_2^{-\frac{1}{2}\times 1}$ gives $\begin{pmatrix} 4 & 2 \\ 0 & 9 \end{pmatrix}$
so factor out diagonal elements, and use inverse of the row operation for L:

$A = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$.