MATH 417 **Final Exam**
Instructor: D. Cabrera **Practice Problems**

Contour Integrals

1. Evaluate $\int_C \frac{e^{\pi z}}{z^2+1}\,dz$ where C is the circle $|z| = 2$ oriented counterclockwise.

2. Evaluate $\int_C \frac{1}{(z+5)(z-1)^5}\,dz$ where C is the circle $|z| = 4$ oriented counterclockwise.

3. Evaluate $\int_C \frac{1}{\sin z}\,dz$ where C is the rectangle with vertices at $-1-i$, $8-i$, $8+i$, and $-1+i$ oriented counterclockwise.

4. Evaluate $\int_D z^8 e^{1/z}\,dz$ where D is the unit circle oriented counterclockwise.

5. Evaluate $\int_C \frac{z^2+1}{\cos z}\,dz$ where C is the circle $\left|z - \frac{\pi}{2}\right| = 1$ oriented counterclockwise.

6. Evaluate $\int_C \frac{(\operatorname{Log} z)^2}{z^2+9}\,dz$ where C is the circle $|z-3i| = 1$ oriented counterclockwise.

7. Evaluate $\int_C \frac{\sin z}{z^2(2z-i)^3}\,dz$ where C is the circle $|z| = 2$ oriented counterclockwise.

8. Evaluate $\int_C \frac{e^z + \cos z}{z^2(z-i)^3}\,dz$ where C is the circle $|z-i| = \frac{1}{2}$ oriented counterclockwise.

Classifying Singular Points

9. Find all isolated singularities of each function below. Then classify each singularity as being a removable singularity, a pole of order N, or an essential singularity.

(a) $\dfrac{1}{z^3+1}$ (b) $\dfrac{z^2}{1-\cos z}$ (c) $\dfrac{1}{\sin\frac{1}{z}}$ (d) $\dfrac{z^3+1}{z^2(z+1)}$ (e) $z^3e^{1/z}$

(f) $\dfrac{z^2}{e^z-1}$ (g) $\dfrac{1}{\sin(z-2)}$ (h) $\dfrac{\operatorname{Log}(z+1)}{z}$

Taylor, Laurent Series

10. Find the Laurent Series for the function $f(z) = \dfrac{1}{z^3}\sin\left(\dfrac{1}{z^2}\right)$ on the region $0 < |z| < \infty$.

11. Find the Laurent Series for the function $f(z) = \dfrac{2}{(z-4)(z-6)}$

(a) on the annulus $4 < |z| < 6$

(b) on the domain $|z| > 6$

12. Find the Taylor Series for $\dfrac{1}{z}$ about $z = 1$.

13. Find the Laurent Series of $f(z) = \dfrac{z+5}{z^2-2z-3}$ centered at the origin for $|z| < 1$ and for $|z| > 3$.

14. Determine all possible Taylor and Laurent series expansions for $f(z) = \dfrac{1}{z}$ about $z = -2$ and state their regions of validity.

15. Determine all possible Taylor and Laurent series expansions for $f(z) = \dfrac{1}{z(z-2)}$ about $z = 0$ and state their regions of validity.

16. Determine all possible Taylor and Laurent series expansions for $f(z) = \dfrac{1}{z^2+iz+2}$ about $z = i$ and state their regions of validity.

Improper Integrals

17. Compute the integral $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^2}$.

18. Compute the integral $\displaystyle\int_{-\infty}^\infty \frac{dx}{4x^2+2x+1}$.

19. Compute the integral $\displaystyle\int_0^\infty \frac{(\ln x)^2}{x^2+1}\,dx$.

20. Show that $\displaystyle\int_0^\infty \frac{x^2}{x^4+x^2+1}\,dx = \frac{\pi}{\sqrt{3}}$.

21. Compute the integral $\displaystyle\int_{-\infty}^\infty \frac{\cos x}{x^2+a^2}\,dx, \quad a>0$.

22. Show that $\displaystyle\int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{2ab(a+b)}$ where $a, b > 0$ and $a \neq b$.

23. Compute the value of $\displaystyle\int_{-\infty}^\infty \frac{x^2}{(1+x^2)(4+x^2)}\,dx$.

24. Show that $\displaystyle\int_0^\infty \frac{\ln x}{1+x^4}\,dx = \frac{\pi^2}{8\sqrt{2}}$.

Trigonometric Integrals

25. Compute the integral $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+\cos\theta}, \quad a>1$.

26. Determine the value of $\displaystyle\int_0^{2\pi} \frac{d\theta}{4+3\cos\theta}$.

27. Compute the integral $\displaystyle\int_{-\pi}^{\pi} \sin^2\theta\,d\theta$ using a contour integral.

28. Evaluate the integral $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}, \quad 0<a<1.$

29. Show that $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}\, d\theta = \frac{\pi}{4}.$

30. Evaluate the integral $\displaystyle\int_0^{\pi} \cos^4\theta\, d\theta$ using a contour integral.

31. Compute the integral $\displaystyle\int_0^{\pi} \frac{d\theta}{1+\cos^2\theta}.$

32. Show that $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+2\sin 2\theta} = \frac{2\pi}{\sqrt{3}}.$

Inverse Laplace Transform

33. Use the formula for the Inverse Laplace Transform to evaluate the inverse of each of the following functions

(a) $\dfrac{1}{s^2}$ (b) $\dfrac{1}{s^2+4}$ (c) $\dfrac{1}{(s-1)^2+9}$ (d) $\dfrac{s}{(s^2+1)^2}$

Rouché's Theorem

34. Show that z^5+2z^2+3z+2 has five zeros, counting multiplicities, inside the circle $|z|=2$.

35. Determine the number of zeros (counting multiplicities) of z^4+7z+5 that lie inside the circle $|z|=1$.

36. Find the number of zeros (counting multiplicities) of the polynomial $2z^8+3z^5-9z^3+2$ inside the annulus $1<|z|<2$.

37. Determine the number of zeros (counting multiplicities) of the polynomial $z^5+3z^3+z^2+1$ inside the circle $|z|=2$.

Möbius Transformations

38. Find a Möbius transformation $f(z)$ that maps the points 1, 2, and i to the points i, 1, and 2, respectively. That is, $f(1) = i$, $f(2) = 1$, and $f(i) = 2$.

39. Find a Möbius transformation $f(z)$ that maps the points $-i$, 0, and i to the points -1, i, and 1, respectively.

40. Find a Möbius transformation $f(z)$ that maps the points 1, 0, and -1 to the points i, ∞, and 1, respectively.