MATH 417 **Homework 1**
Instructor: D. Cabrera **Due June 18**

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1. Find the modulus and conjugate of each complex number below.

   (a) $-2+i$

   (b) $(2+i)(4+3i)$

   (c) $\dfrac{3-i}{\sqrt{2}+3i}$

   Solution:

   (a) For the complex number $z=-2+i$, the modulus is

   $$|z|=\sqrt{(-2)^2+1^2}=\sqrt{5}$$

   and the conjugate is

   $$\bar{z}=-2-i$$

   (b) For the complex number $z=(2+i)(4+3i)=5+10i$, the modulus is

   $$|z|=\sqrt{5^2+10^2}=5\sqrt{5}$$

   and the conjugate is

   $$\bar{z}=5-10i$$

   (c) For the complex number $z=\dfrac{3-i}{\sqrt{2}+3i}$, the modulus is

   $$|z|=\frac{|3-i|}{|\sqrt{2}+3i|}=\frac{\sqrt{3^2+(-1)^2}}{\sqrt{(\sqrt{2})^2+3^2}}=\frac{\sqrt{10}}{\sqrt{11}}$$

   and the conjugate is

   $$\bar{z}=\frac{\overline{3-i}}{\overline{\sqrt{2}+3i}}=\frac{3+i}{\sqrt{2}-3i}=\frac{(3+i)(\sqrt{2}+3i)}{(\sqrt{2}-3i)(\sqrt{2}+3i)}=\frac{3\sqrt{2}-3}{11}+i\,\frac{\sqrt{2}+9}{11}$$

2. Express each complex number below in exponential form. In each case, use the principal argument of the number.

   (a) $2i$

   (b) $1+i$

   (c) $-2+i\sqrt{12}$

Solution:

(a) The modulus of $z = 2i$ is

$$|z| = \sqrt{0^2 + 2^2} = 2$$

The principal argument Θ is found from the equations

$$\cos\Theta = \frac{x}{|z|} = \frac{0}{2} = 0$$
$$\sin\Theta = \frac{y}{|z|} = \frac{2}{2} = 1$$

and is $\Theta = \frac{\pi}{2}$. Therefore, the exponential form of z is

$$\boxed{z = 2i = 2e^{i(\pi/2)}}$$

(b) The modulus of $z = 1 + i$ is

$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

The principal argument Θ is found from the equations

$$\cos\Theta = \frac{x}{|z|} = \frac{1}{\sqrt{2}}$$
$$\sin\Theta = \frac{y}{|z|} = \frac{1}{\sqrt{2}}$$

and is $\Theta = \frac{\pi}{4}$. Therefore, the exponential form of z is

$$\boxed{z = 1 + i = \sqrt{2}e^{i(\pi/4)}}$$

(c) The modulus of $z = -2 + i\sqrt{12}$ is

$$|z| = \sqrt{(-2)^2 + (\sqrt{12})^2} = \sqrt{4 + 12} = 4$$

The principal argument Θ is found from the equations

$$\cos\Theta = \frac{x}{|z|} = \frac{-2}{4} = -\frac{1}{2}$$
$$\sin\Theta = \frac{y}{|z|} = \frac{\sqrt{12}}{4} = \frac{\sqrt{3}}{2}$$

and is $\Theta = \frac{2\pi}{3}$. Therefore, the exponential form of z is

$$\boxed{z = -2 + i\sqrt{12} = 4e^{i(2\pi/3)}}$$

3. Use DeMoivre's Theorem to expand $(1+i)^6$. Write your answer in the form $a+bi$.

Solution: The modulus and principal argument of $1+i$ are $|z| = r = \sqrt{2}$ and $\Theta = \dfrac{\pi}{4}$. Then, using DeMoivre's Theorem we have

$$\begin{aligned} z^6 &= r^6\left[\cos(6\Theta) + i\sin(6\Theta)\right] \\ (1+i)^6 &= \left(\sqrt{2}\right)^6\left[\cos\left(6\cdot\frac{\pi}{4}\right) + i\sin\left(6\cdot\frac{\pi}{4}\right)\right] \\ &= 8(0-i) \\ &= \boxed{-8i} \end{aligned}$$

4. Show that $\overline{e^{i\theta}} = e^{-i\theta}$.

Solution: Starting with the left hand side we have

$$\overline{e^{i\theta}} = \overline{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta$$

Now use the negative angle identities

$$\cos(-\theta) = \cos\theta, \quad \sin(-\theta) = -\sin\theta$$

to write $\cos\theta - i\sin\theta$ as

$$\overline{e^{i\theta}} = \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta) = e^{i(-\theta)} = e^{-i\theta}$$

5. Find all solutions to $z^4 = -16$.

Solution: The modulus and principal argument of -16 are $|z| = r = 16$ and $\Theta = \pi$, respectively. The fourth roots of -16 are given by the formula

$$z = r^{1/4}\left[\cos\left(\frac{\Theta}{4} + \frac{2k\pi}{4}\right) + i\sin\left(\frac{\Theta}{4} + \frac{2k\pi}{4}\right)\right], \quad \text{for } k = 0, 1, 2, 3$$

The roots are then

$$z_1 = 16^{1/4}\left[\cos\left(\frac{\pi}{4} + \frac{2(0)\pi}{4}\right) + i\sin\left(\frac{\pi}{4} + \frac{2(0)\pi}{4}\right)\right] = 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \sqrt{2} + i\sqrt{2}$$

$$z_2 = 16^{1/4}\left[\cos\left(\frac{\pi}{4} + \frac{2(1)\pi}{4}\right) + i\sin\left(\frac{\pi}{4} + \frac{2(1)\pi}{4}\right)\right] = 2\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = -\sqrt{2} + i\sqrt{2}$$

$$z_3 = 16^{1/4}\left[\cos\left(\frac{\pi}{4} + \frac{2(2)\pi}{4}\right) + i\sin\left(\frac{\pi}{4} + \frac{2(2)\pi}{4}\right)\right] = 2\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = -\sqrt{2} - i\sqrt{2}$$

$$z_4 = 16^{1/4}\left[\cos\left(\frac{\pi}{4} + \frac{2(3)\pi}{4}\right) + i\sin\left(\frac{\pi}{4} + \frac{2(3)\pi}{4}\right)\right] = 2\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = \sqrt{2} - i\sqrt{2}$$

6. Solve the equation

$$z^{4/3} + 2i = 0$$

for z and plot the roots in the complex plane.

Solution: The solutions to the equation are given by

$$z = (-2i)^{3/4} = \left[(-2i)^3\right]^{1/4} = (8i)^{1/4}$$

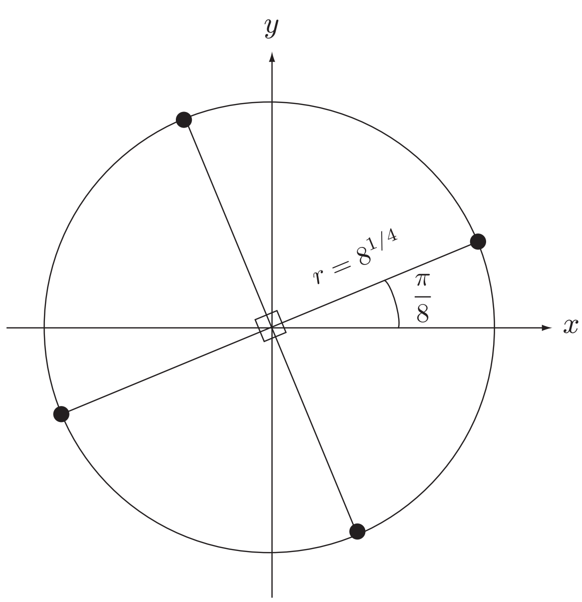
The modulus and principal argument of $8i$ are $|z| = r = 8$ and $\Theta = \dfrac{\pi}{2}$, respectively. The fourth roots of $8i$ are given by the formula

$$z = r^{1/4}\left[\cos\left(\frac{\Theta}{4} + \frac{2k\pi}{4}\right) + i\sin\left(\frac{\Theta}{4} + \frac{2k\pi}{4}\right)\right], \quad \text{for} \quad k = 0, 1, 2, 3$$

The roots are then

$$z_1 = 8^{1/4}\left[\cos\left(\frac{\pi/2}{4} + \frac{2(0)\pi}{4}\right) + i\sin\left(\frac{\pi}{4} + \frac{2(0)\pi}{4}\right)\right] = \sqrt[4]{8}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$$
$$z_2 = 8^{1/4}\left[\cos\left(\frac{\pi/2}{4} + \frac{2(1)\pi}{4}\right) + i\sin\left(\frac{\pi}{4} + \frac{2(1)\pi}{4}\right)\right] = \sqrt[4]{8}\left(\cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8}\right)$$
$$z_3 = 8^{1/4}\left[\cos\left(\frac{\pi/2}{4} + \frac{2(2)\pi}{4}\right) + i\sin\left(\frac{\pi}{4} + \frac{2(2)\pi}{4}\right)\right] = \sqrt[4]{8}\left(\cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8}\right)$$
$$z_4 = 8^{1/4}\left[\cos\left(\frac{\pi/2}{4} + \frac{2(3)\pi}{4}\right) + i\sin\left(\frac{\pi}{4} + \frac{2(3)\pi}{4}\right)\right] = \sqrt[4]{8}\left(\cos\frac{13\pi}{8} + i\sin\frac{13\pi}{8}\right)$$

and lie on a circle of radius $\sqrt[4]{8}$ centered at the origin, $\dfrac{\pi}{2}$ radians apart.



7. Write the function $f(z) = z^3 + z + 1$ in the form $f(x, y) = u(x, y) + i\,v(x, y)$.

Solution: To write $f(z)$ in terms of x and y we substitute $z = x + iy$ and simplify.

$$\begin{aligned} f(z) &= (x + iy)^3 + (x + iy) + 1 \\ &= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 + x + iy + 1 \\ &= x^3 + i(3x^2y) - 3xy^2 - i(y^3) + x + i(y) + 1 \\ &= \boxed{(x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y)} \end{aligned}$$

8. Suppose that $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$, where $z = x + iy$. Use the expressions

$$x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2i}$$

to write $f(z)$ in terms of z and simplify the result.

Solution: Substituting the above expressions into $f(z)$ and simplifying we get

$$\begin{aligned} f(z) &= \left(\frac{z + \bar{z}}{2}\right)^2 - \left(\frac{z - \bar{z}}{2i}\right)^2 - 2\left(\frac{z - \bar{z}}{2i}\right) + i\left[2\left(\frac{z + \bar{z}}{2}\right) - 2\left(\frac{z + \bar{z}}{2}\right)\left(\frac{z - \bar{z}}{2i}\right)\right] \\ &= \frac{1}{4}(z^2 + 2z\bar{z} + \bar{z}^2) + \frac{1}{4}(z^2 - 2z\bar{z} + \bar{z}^2) + i(z - \bar{z}) + i\left[z + \bar{z} + i\left(\frac{1}{2}z^2 - \frac{1}{2}\bar{z}^2\right)\right] \\ &= \frac{1}{2}z^2 + \frac{1}{2}\bar{z}^2 - \frac{1}{2}z^2 + \frac{1}{2}\bar{z}^2 + i(z - \bar{z} + z + \bar{z}) \\ &= \boxed{\bar{z}^2 + i(2z)} \end{aligned}$$

9. Find the image of the semi-infinite strip $x \geq 0$, $0 \leq y \leq \pi$ under the transformation $w = e^z$ and label corresponding portions of the boundaries.

Solution: We did this in class.