Math 180, Exam 2, Spring 2012
Problem 1 Solution

1. Compute $f'(x)$ for the following functions:

(a) $f(x) = 6^x$

(b) $f(x) = x^2 \ln\left(\frac{x}{5}\right)$

(c) $f(x) = \arcsin(e^{2x})$.

Solution:

(a) Using the rule $(b^x)' = \ln(b)\, b^x$ we get

$$\boxed{f'(x) = \ln(6)\, 6^x}.$$

(b) Using the Product, Chain Rule, and the rule $[\ln(x)]' = \frac{1}{x}$ we get

$$\begin{aligned}
f'(x) &= x^2 \left[\ln\left(\frac{x}{5}\right)\right]' + \left(x^2\right)' \ln\left(\frac{x}{5}\right), \\
f'(x) &= x^2 \cdot \frac{1}{\frac{x}{5}} \cdot \left(\frac{x}{5}\right)' + 2x \ln\left(\frac{x}{5}\right), \\
f'(x) &= x^2 \cdot \frac{1}{\frac{x}{5}} \cdot \frac{1}{5} + 2x \ln\left(\frac{x}{5}\right), \\
f'(x) &= \boxed{x + 2x \ln\left(\frac{x}{5}\right)}.
\end{aligned}$$

(c) Using the Chain Rule and the rule $[\arcsin(x)]' = \frac{1}{\sqrt{1-x^2}}$ we get

$$\begin{aligned}
f'(x) &= \frac{1}{\sqrt{1-(e^{2x})^2}} \cdot \left(e^{2x}\right)', \\
f'(x) &= \boxed{\frac{1}{\sqrt{1-e^{4x}}} \cdot 2e^{2x}}.
\end{aligned}$$

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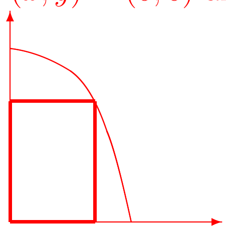
Problem 2 Solution

2. Find the largest possible area of a rectangle having its lower left corner at the origin $(x, y) = (0, 0)$ and its upper right corner on the parabola $y = 9 - x^2$.



Solution: Since the lower left corner of the rectangle is at the origin and the upper right corner is on the curve $y = 9 - x^2$, we know that the upper right corner has coordinates (x, y) which represent the width and height of the rectangle, respectively. Therefore, the area function is

$$A(x) = xy = x(9 - x^2) = 9x - x^3$$

The domain of this function is $[0, 3]$, where $x = 3$ is an x-intercept of $y = 9-x^2$. The critical points of $A(x)$ in its domain satisfy $A'(x) = 0$.

$$\begin{aligned} A'(x) &= 0, \\ 9 - 3x^2 &= 0, \\ x^2 &= 3, \\ x &= \sqrt{3}. \end{aligned}$$

Note that $x = -\sqrt{3}$ is a solution to $A'(x) = 0$ but is not in the domain of $A(x)$. To determine the absolute maximum value of $A(x)$, we evaluate the function at $x = \sqrt{3}$ and at the endpoints of the domain.

$$A\left(\sqrt{3}\right) = 6\sqrt{3}, \quad A(0) = 0, \quad A(3) = 0$$

Clearly, the largest value is $6\sqrt{3}$ and, thus, corresponds to the absolute maximum.

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Problem 3 Solution

3. Determine if $y = -2x^3 + 3x^2 + 36x - 7$ is concave up, concave down, or neither concave up nor concave down on the interval $(0, 2)$. Justify your answer.

Solution: The first two derivatives of y are

$$y' = -6x^2 + 6x + 36,$$
$$y'' = -12x + 6.$$

We notice that $y''(\frac{1}{4}) = 3$ and $y''(1) = -6$. Thus, since y'' is neither positive nor negative for all values of x in $(0, 2)$, the function is neither concave up nor concave down on $(0, 2)$.

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Problem 4 Solution

4. Consider the curve $\mathcal{C}$ in the plane given by the equation $x\ln(y) + y\ln(x) = 1$.

(a) Find the slope of the tangent line to $\mathcal{C}$ at the point $(x, y) = (e, 1)$.

(b) Find an equation for the tangent line to $\mathcal{C}$ at the point $(x, y) = (e, 1)$.

Solution:

(a) The derivative $\frac{dy}{dx}$, which represents the slope of the tangent line, must be found using implicit differentiation. The derivatives of the terms on the left hand side of the given equation are determined using the Product and Chain Rules.

$$\begin{aligned}
\frac{d}{dx}x\ln(y) &= x\frac{d}{dx}\ln(y) + \ln(y)\frac{d}{dx}x & \qquad \frac{d}{dx}y\ln(x) &= y\frac{d}{dx}\ln(x) + \ln(x)\frac{d}{dx}y,\\
\frac{d}{dx}x\ln(y) &= x\cdot\frac{1}{y}\frac{dy}{dx} + \ln(y) & \qquad \frac{d}{dx}y\ln(x) &= y\cdot\frac{1}{x} + \ln(x)\frac{dy}{dx},\\
\frac{d}{dx}x\ln(y) &= \frac{x}{y}\frac{dy}{dx} + \ln(x) & \qquad \frac{d}{dx}y\ln(x) &= \frac{y}{x} + \ln(x)\frac{dy}{dx}
\end{aligned}$$

Differentiating both sides of the given equation and solving for $\frac{dy}{dx}$ we get

$$\begin{aligned}
\frac{d}{dx}x\ln(y) + \frac{d}{dx}y\ln(x) &= \frac{d}{dx}1,\\
\frac{x}{y}\frac{dy}{dx} + \ln(y) + \frac{y}{x} + \ln(x)\frac{dy}{dx} &= 0,\\
\frac{x}{y}\frac{dy}{dx} + \ln(x)\frac{dy}{dx} &= -\ln(y) - \frac{y}{x},\\
\frac{dy}{dx}\left(\frac{x}{y} + \ln(x)\right) &= -\ln(y) - \frac{y}{x},\\
\frac{dy}{dx} &= \frac{-\ln(y) - \frac{y}{x}}{\frac{x}{y} + \ln(x)}.
\end{aligned}$$

At the point $(e, 1)$ we have

$$\begin{aligned}
\left.\frac{dy}{dx}\right|_{(e,1)} &= \frac{-\ln(1) - \frac{1}{e}}{\frac{e}{1} + \ln(e)},\\
\left.\frac{dy}{dx}\right|_{(e,1)} &= \frac{-0 - \frac{1}{e}}{e+1},\\
\left.\frac{dy}{dx}\right|_{(e,1)} &= \boxed{-\frac{1}{e(e+1)}}.
\end{aligned}$$

(b) From part (a) we know that the slope of the tangent line at $(e, 1)$ is $-\frac{1}{e(e+1)}$. Therefore, the tangent line in point-slope form is

$$\boxed{y - 1 = -\frac{1}{e(e+1)}(x - e)}$$

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Problem 5 Solution

5. Let $f(x)$ be the function defined by $f(x) = x + \cos(x)$.

(a) Find and classify the critical points of $f(x)$ on the interval $[0, 4\pi]$.

(b) Find the absolute minimum and and absolute maximum of $f(x)$ on the same interval.

Solution:

(a) The function $f(x)$ is continuous everywhere. Therefore, the only critical points of the function will be solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0, \\ 1 - \sin(x) &= 0, \\ \sin(x) &= 1, \\ x &= \frac{\pi}{2}, \frac{5\pi}{2}. \end{aligned}$$

Note that the equation $\sin(x) = 1$ has infinitely many solutions but there are only two on the interval $[0, 4\pi]$. We use the First Derivative Test to classify the critical points.

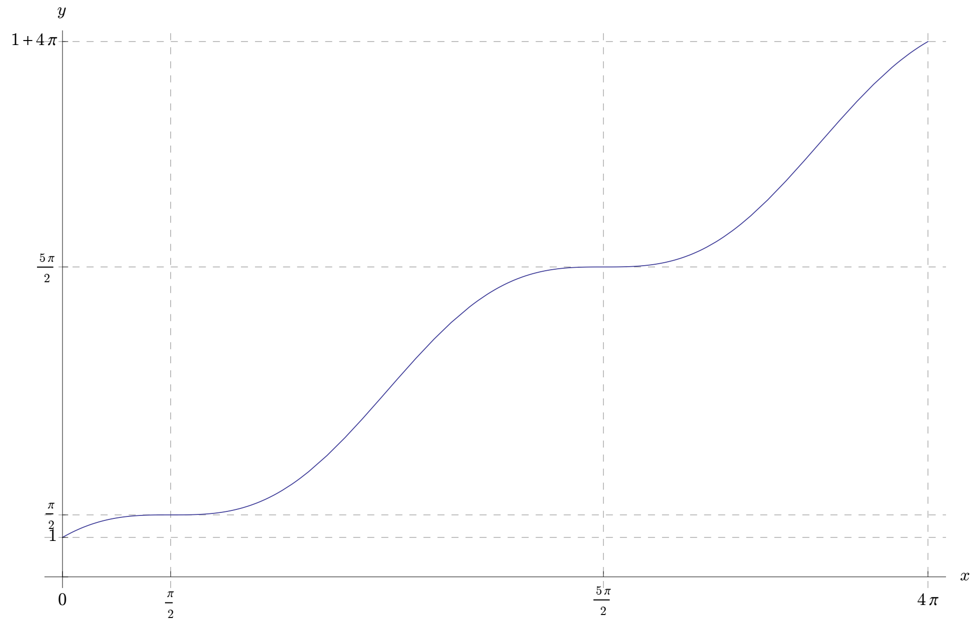
Interval	**Test Point**, c	$f'(c)$	**Sign of** $f'(c)$
$(0, \frac{\pi}{2})$	$\frac{\pi}{6}$	$f'(\frac{\pi}{6}) = \frac{1}{2}$	$+$
$(\frac{\pi}{2}, \frac{5\pi}{2})$	π	$f'(\pi) = 1$	$+$
$(\frac{5\pi}{2}, 4\pi)$	3π	$f'(3\pi) = 1$	$+$

From the table we see that there are no sign changes in $f'(x)$ across either of the critical points. Therefore, the critical points correspond to neither local maxima nor local minima.

(b) To find the absolute extrema we evaluate $f(x)$ at the critical points and at the endpoints of the interval $[0, 4\pi]$.

$$f(0) = 1, \quad f(\tfrac{\pi}{2}) = \tfrac{\pi}{2}, \quad f(\tfrac{5\pi}{2}) = \tfrac{5\pi}{2}, \quad f(4\pi) = 4\pi + 1$$

The largest of the above function values is $5\pi + 1$ and the smallest is 1. These values are the absolute maximum and minimum, respectively.


y
$1+4\pi$
$\frac{5\pi}{2}$
$\frac{\pi}{2}$
1
0
$\frac{\pi}{2}$
$\frac{5\pi}{2}$
4π
x