Math 180, Final Exam, Fall 2009
Problem 1 Solution

1. Differentiate with respect to x. Write your answers showing the use of the appropriate techniques. Do **not** simplify.

(a) $(x^2+1)\sin(x)$ (b) $\dfrac{5x-3}{x^2+x+1}$ (c) e^{x^2+3}

Solution:

(a) Use the Product Rule.

$$\begin{aligned}[(x^2+1)\sin(x)]' &= (x^2+1)(\sin(x))' + (x^2+1)'\sin(x) \\ &= \boxed{(x^2+1)\cos x + 2x\sin x}\end{aligned}$$

(b) Use the Quotient Rule.

$$\begin{aligned}\left(\frac{5x-3}{x^2+x+1}\right)' &= \frac{(x^2+x+1)(5x-3)' - (5x-3)(x^2+x+1)'}{(x^2+x+1)^2} \\ &= \boxed{\frac{(x^2+x+1)(5) - (5x-3)(2x+1)}{(x^2+x+1)^2}}\end{aligned}$$

(c) Use the Chain Rule.

$$\begin{aligned}(e^{x^2+3})' &= e^{x^2+3}\cdot(x^2+3)' \\ &= \boxed{e^{x^2+3}\cdot(2x)}\end{aligned}$$

Math 180, Final Exam, Fall 2009
Problem 2 Solution

2.

(a) Write a definite integral that gives the area of the region between $x = 0$ and $x = 2$, above the x-axis, and below the curve $y = 4x - x^3$.

(b) Evaluate this integral to find the exact area.

Solution:

(a) The area of the region is given by the integral:

$$\int_0^2 (4x - x^3)\, dx$$

(b) We use FTC I to evaluate the integral.

$$\begin{aligned}\int_0^2 (4x - x^3)\, dx &= 2x^2 - \frac{x^4}{4}\bigg|_0^2 \\ &= \left(2(2)^2 - \frac{2^4}{4}\right) - \left(2(0)^2 - \frac{0^4}{4}\right) \\ &= \boxed{4}\end{aligned}$$

Math 180, Final Exam, Fall 2009
Problem 3 Solution

3. Find

$$\lim_{x\to 0} \frac{1-\cos x}{x^2}$$

Explain how you obtain your answer.

Solution: Upon substituting $x = 0$ into the function we find that

$$\frac{1-\cos x}{x^2} = \frac{1-\cos 0}{0^2} = \frac{0}{0}$$

which is indeterminate. We resolve this indeterminacy by using L'Hôpital's Rule.

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos x}{x^2} &\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{(1-\cos x)'}{(x^2)'} \\
&= \lim_{x\to 0} \frac{\sin x}{2x} \\
&= \frac{1}{2} \lim_{x\to 0} \frac{\sin x}{x} \\
&= \boxed{\frac{1}{2}}
\end{aligned}$$

Note: We used the fact that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ to evaluate the limit.

Math 180, Final Exam, Fall 2009
Problem 4 Solution

4. Find an antiderivative for $f(x) = x^3 + x^{-2} + x^{2/3}$, that is, find $\int \left(x^3 + x^{-2} + x^{2/3}\right)\, dx$.

Solution: An antiderivative for $f(x)$ is:

$$\begin{aligned}\int \left(x^3 + x^{-2} + x^{2/3}\right)\, dx &= \frac{x^{3+1}}{3+1} + \frac{x^{-2+1}}{-2+1} + \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} + C \\ &= \boxed{\frac{1}{4}x^4 - \frac{1}{x} + \frac{3}{5}x^{5/3} + C}\end{aligned}$$

Math 180, Final Exam, Fall 2009
Problem 5 Solution

5.

(a) Find the critical points of the function $f(x) = \dfrac{x-1}{x^2-x+1}$ and classify each critical point as a local minimum, local maximum, or neither.

(b) Find the exact values of $f(x)$ at the critical points.

(c) The graph $y = f(x)$ has a horizontal asymptote $y = L$. What is the value of L?

(d) Sketch the graph $y = f(x)$ using your answers to (b) and (c).

Solution:

(a),(b) The critical points of $f(x)$ are the values of x for which either $f'(x)$ does not exist or $f'(x) = 0$.

$$\begin{aligned}
f'(x) &= 0 \\
\left(\frac{x-1}{x^2-x+1}\right)' &= 0 \\
\frac{(x^2-x+1)(x-1)' - (x-1)(x^2-x+1)'}{(x^2-x+1)^2} &= 0 \\
\frac{(x^2-x+1)(1) - (x-1)(2x-1)}{(x^2-x+1)^2} &= 0 \\
\frac{-x^2+2x}{(x^2-x+1)^2} &= 0 \\
-x^2+2x &= 0 \\
x(-x+2) &= 0 \\
x = 0,\ x &= 2
\end{aligned}$$

Thus, $\boxed{x=0}$ and $\boxed{x=2}$ are the critical points of f. (Note: $x^2-x+1 > 0$ for all x.) We now use the Second Derivative Test to classify the critical points. The second derivative $f''(x)$ is:

$$\begin{aligned}
f''(x) &= \frac{(x^2-x+1)^2(-x^2+2x)' - (-x^2+2x)[(x^2-x+1)^2]'}{(x^2-x+1)^4} \\
f''(x) &= \frac{(x^2-x+1)^2(-2x+2) - (-x^2+2x)[2(x^2-x+1)(2x-1)]}{(x^2-x+1)^4} \\
f''(x) &= \frac{(x^2-x+1)(-2x+2) - 2(-x^2+2x)(2x-1)}{(x^2-x+1)^3} \\
f''(x) &= \frac{-2x^3+4x^2-4x+2+4x^3-10x^2+4x}{(x^2-x+1)^3} \\
f''(x) &= \frac{2x^3-6x^2+2}{(x^2-x+1)^3}
\end{aligned}$$

At the critical points we have:

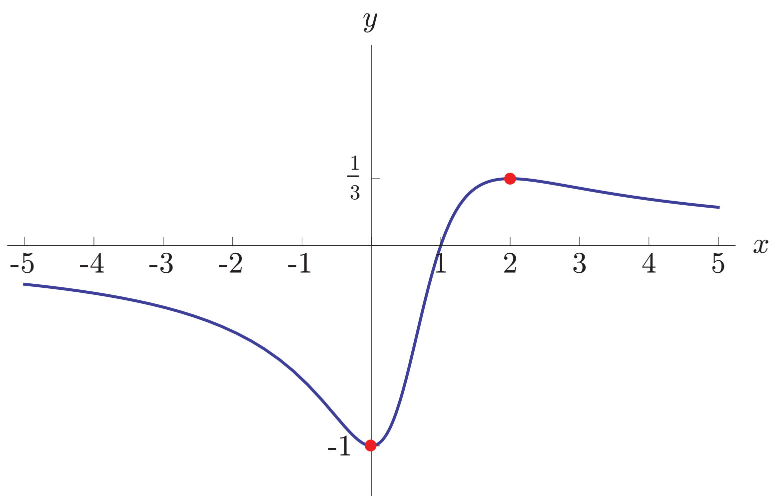
$$f''(0) = 2, \quad f''(2) = -\frac{2}{9}$$

Since $f''(0) > 0$, the Second Derivative Test implies that the point $f(0) = -1$ is a local minimum. Since $f''(2) < 0$, the Second Derivative Test implies that the point $f(2) = \frac{1}{3}$ is a local maximum.

(c) The horizontal asymptote is:

$$y = \lim_{x\to\infty} \frac{x-1}{x^2 - x + 1} = 0$$

(d) The graph of $y = f(x)$ is shown below:



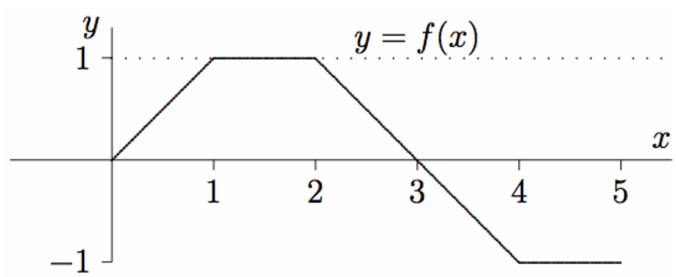
Math 180, Final Exam, Fall 2009
Problem 7 Solution

6. The graph of a function $f(x)$ is shown below.

(a) Find $\displaystyle\int_0^3 f(x)\,dx$.

(b) Find $\displaystyle\int_0^5 f(x)\,dx$.

(c) If $F(x) = \displaystyle\int_0^x f(t)\,dt$, find $F'(2)$.



Solution:

(a) The value of $\displaystyle\int_0^3 f(x)\,dx$ is the signed area between the graph of $y = f(x)$ and the x-axis on the interval $[0,3]$. We use the additivity of integrals to break the integral down as follows:

$$\int_0^3 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx$$

The reason for this is that the regions between $y = f(x)$ and the x-axis on the intervals $[0,1]$, $[1,2]$, and $[2,3]$ are either triangles or rectangles. The signed area is then:

$$\begin{aligned}\int_0^3 f(x)\,dx &= \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx\\ &= \frac{1}{2}(1)(1) + (1)(1) + \frac{1}{2}(1)(1)\\ &= \boxed{2}\end{aligned}$$

(b) The value of $\displaystyle\int_0^5 f(x)\,dx$ is the signed area between the graph of $y = f(x)$ and the x-axis on the interval $[0,5]$. We use the additivity of integrals to break the integral down as follows:

$$\int_0^5 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \int_3^4 f(x)\,dx + \int_4^5 f(x)\,dx$$

The reason for this is that the regions between $y = f(x)$ and the x-axis on the intervals $[0,1]$, $[1,2]$, $[2,3]$, $[3,4]$, and $[4,5]$ are either triangles or rectangles. The signed area is then:

$$\begin{aligned}
\int_0^5 f(x)\,dx &= \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \int_3^4 f(x)\,dx + \int_4^5 f(x)\,dx \\
&= \frac{1}{2}(1)(1) + (1)(1) + \frac{1}{2}(1)(1) - \frac{1}{2}(1)(1) - (1)(1) \\
&= \boxed{\frac{1}{2}}
\end{aligned}$$

(c) Using the Fundamental Theorem of Calculus, Part II we know that $F'(x) = f(x)$. Then the value of $F'(2)$ is $f(2)$ which is the y-coordinate of the point on the graph when $x = 2$. From the graph we see that $F'(2) = f(2) = \boxed{1}$.

Math 180, Final Exam, Fall 2009
Problem 7 Solution

7. Evaluate the integral $\int \sin^3 \cos x \, dx$.

Solution: We use the substitution $u = \sin x$, $du = \cos x \, dx$. Making the substitutions and evaluating the integral we get:

$$\begin{aligned}\int \sin^3 x \cos x \, dx &= \int u^3 \, du \\ &= \frac{u^4}{4} + C \\ &= \boxed{\frac{\sin^4 x}{4} + C}\end{aligned}$$

Math 180, Final Exam, Fall 2009
Problem 8 Solution

8. Evaluate the integral $\displaystyle\int \frac{e^x}{1+e^x}\,dx$.

Solution: We use the substitution $u = 1 + e^x$, $du = e^x\,dx$. Making the substitutions and evaluating the integral we get:

$$\begin{aligned}
\int \frac{e^x}{1+e^x}\,dx &= \int \frac{1}{u}\,du \\
&= \ln|u| + C \\
&= \ln|e^x + 1| + C \\
&= \boxed{\ln(e^x + 1) + C}
\end{aligned}$$

Note: We don't need the absolute values because $e^x + 1$ is always positive.

Math 180, Final Exam, Fall 2009
Problem 9 Solution

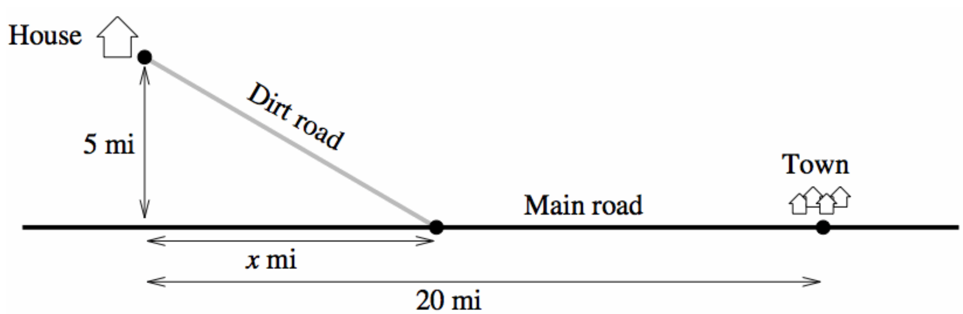
9. A house lies 5 miles north of a paved road that runs east-west. The nearest town lies 20 miles east on the main road. The owner of the house wants to build a dirt road from the house to the main road that will meet the main road at point x miles east of the point due south of the house. A car can travel 30 miles per hour on the dirt road and 50 miles per hour on the main road. The goal of this problem is to find the value of x that will minimize the travel time from the house to the town.

(a) What is the distance from the house, at $(0, 5)$, to the point $(x, 0)$ on the main road?

(b) What is the distance from $(x, 0)$ to town, $(20, 0)$?

(c) What is the total travel time $T(x)$ for the trip from the house to the town as a function of x? Recall that time=distance/speed.

(d) Find the value of x minimizing $T(x)$.



Solution:

(a) Let D be the distance from $(0, 5)$ to $(x, 0)$. Then, using the distance formula, we have:

$$D = \sqrt{(x-0)^2 + (0-5)^2} = \sqrt{x^2 + 25}$$

(b) Let d be the distance from $(x, 0)$ to $(20, 0)$. Then the distance is:

$$d = 20 - x$$

(c) The total travel time $T(x)$ is:

$$T(x) = \text{Dirt road} + \text{Main road}$$
$$T(x) = \frac{\sqrt{x^2 + 25}}{30} + \frac{20 - x}{50}$$

The domain of $T(x)$ is $[0, 20]$.

(d) The critical points of $T(x)$ are the solutions to $T'(x) = 0$.

$$\begin{aligned}
T'(x) &= 0 \\
\left(\frac{\sqrt{x^2+25}}{30} + \frac{20-x}{50}\right)' &= 0 \\
\frac{1}{30}\cdot\frac{1}{2}(x^2+25)^{-1/2}\cdot(x^2+25)' + \frac{1}{50}\cdot(20-x)' &= 0 \\
\frac{1}{60\sqrt{x^2+25}}\cdot(2x) + \frac{1}{50}\cdot(-1) &= 0 \\
\frac{2x}{60\sqrt{x^2+25}} - \frac{1}{50} &= 0 \\
\frac{10x - 6\sqrt{x^2+25}}{30\sqrt{x^2+25}} &= 0 \\
10x - 6\sqrt{x^2+25} &= 0 \\
3\sqrt{x^2+25} &= 5x \\
\left(3\sqrt{x^2+25}\right)^2 &= (5x)^2 \\
9(x^2+25) &= 25x^2 \\
9x^2 + 225 &= 25x^2 \\
x^2 &= \frac{225}{16} \\
x &= \frac{15}{4}
\end{aligned}$$

The travel time at $x = \frac{15}{4}$ is:

$$T\left(\frac{15}{4}\right) = \frac{\sqrt{(\frac{15}{4})^2+25}}{30} + \frac{20-\frac{15}{4}}{50} = \frac{8}{15}$$

At the endpoints we have:

$$\begin{aligned}
T(0) &= \frac{\sqrt{0^2+25}}{30} + \frac{20-0}{50} = \frac{17}{30} \\
T(20) &= \frac{\sqrt{20^2+25}}{30} + \frac{20-20}{50} = \frac{\sqrt{17}}{6}
\end{aligned}$$

both of which are larger than $\frac{8}{15}$. Thus, $x = \frac{15}{4}$ is the absolute minimum of $T(x)$ on the interval $[0, 20]$.