Math 180, Final Exam, Fall 2013
Problem 1 Solution

1. Compute each limit or explain why it does not exist.

(a) $\displaystyle\lim_{x\to\pi} \frac{\cos^2(x) - 1}{x - \pi}$

(b) $\displaystyle\lim_{x\to 3} \frac{|x-3|}{x-3}$

(c) $\displaystyle\lim_{x\to+\infty} \frac{-3x^2+2x-1}{4+3x+5x^2}$ [Justify your answer using calculus.]

Solution:

(a) This limit is of the form 0/0. Thus, L'Hopital's Rule applies.

$$\lim_{x\to\pi} \frac{\cos^2(x)-1}{x-\pi} = \lim_{x\to\pi} \frac{(\cos^2(x)-1)'}{(x-\pi)'} = \lim_{x\to\pi} \frac{2\cos(x)\cdot(-\sin(x))}{1} = \frac{2\cos(\pi)\cdot(-\sin(\pi))}{1} = 0$$

(b) The function $f(x) = |x-3|/(x-3)$ may be rewritten as

$$f(x) = \begin{cases} 1 & \text{if } x > 3 \\ -1 & \text{if } x < 3 \end{cases}$$

Thus, the one-sided limits are

$$\lim_{x\to 3^+} f(x) = 1, \quad \lim_{x\to 3^-} f(x) = -1$$

Since the one-sided limits are different, the limit does not exist.

(c) This limit is of the form $-\infty/\infty$. Thus, L'Hopital's Rule applies. In fact, it must be applied twice.

$$\lim_{x\to+\infty} \frac{-3x^2+2x-1}{4+3x+5x^2} = \lim_{x\to+\infty} \frac{-6x+2}{3+10x} = \lim_{x\to+\infty} \frac{-6}{10} = \frac{-6}{10} = \frac{-3}{5}$$

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Problem 2 Solution

2. Compute the derivative of each function below. Do not simplify your answer.

   (a) $x^9 e^{2x}$

   (b) $\log_5(4x+2)$

   (c) $\arcsin(1/x)$

Solution:

(a) The Product Rule yields

$$\frac{d}{dx}x^9 e^{2x} = 9x^8 e^{2x} + 2x^9 e^{2x}$$

(b) The logarithm and Chain Rules yield

$$\frac{d}{dx}\log_5(4x+2) = \frac{1}{\ln(5)} \cdot \frac{1}{4x+2} \cdot 4$$

(c) The rule for $\arcsin(x)$ and the Chain Rule yield

$$\frac{d}{dx}\arcsin(1/x) = \frac{1}{\sqrt{1-(1/x)^2}} \cdot \left(-\frac{1}{x^2}\right)$$

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Problem 3 Solution

3. Compute the indefinite integrals.

(a) $\int (2x+1)^3\, dx$

(b) $\int \left(\frac{\ln(x)}{x} + e^{10x} \right) dx$

Solution:

(a) Let $u = 2x+1$. Then $du = 2\, dx$ and $\frac{1}{2}\, du = dx$. The substitution yields

$$\int (2x+1)^3\, dx = \int u^3 \cdot \frac{1}{2}\, du = \frac{1}{2} \int u^3\, du = \frac{1}{2} \cdot \frac{1}{4} u^4 + C = \frac{1}{8}(2x+1)^4 + C$$

(b) The sum rule yields

$$\int \left(\frac{\ln(x)}{x} + e^{10x} \right) dx = \int \frac{\ln(x)}{x}\, dx + \int e^{10x}\, dx$$

Letting $u = \ln(x)$ in the first integral yields $du = \frac{1}{x}\, dx$. Thus,

$$\int \frac{\ln(x)}{x}\, dx + \int e^{10x}\, dx = \int u\, du + \int e^{10x}\, dx = \frac{1}{2} u^2 + \frac{1}{10} e^{10x} + C = \frac{1}{2}(\ln(x))^2 + \frac{1}{10} e^{10x} + C$$

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Problem 4 Solution

4. Compute the definite integrals. Simplify your answers.

(a) $\displaystyle\int_0^1 \frac{5x}{1+x^2}\,dx$

(b) $\displaystyle\int_{-1}^1 \frac{7}{1+x^2}\,dx$

Solution:

(a) Letting $u = 1 + x^2$ we have $du = 2x\,dx$ which yields $\frac{1}{2}\,du = x\,dx$. The limits of integration become $u = 1 + 0^2 = 1$ and $u = 1 + 1^2 + 1 = 2$. The integral becomes:

$$\int_0^1 \frac{5x}{1+x^2}\,dx = \int_1^2 \frac{5}{u}\cdot\frac{1}{2}\,du = \frac{5}{2}\int_1^2 \frac{1}{u}\,du = \frac{5}{2}\Big[\ln|u|\Big]_1^2 = \frac{5}{2}\ln(2)$$

(b) The integrand is an even function. That is, $f(-x) = f(x)$. Therefore, the integral is:

$$\int_{-1}^1 \frac{7}{1+x^2}\,dx = 2\int_0^1 \frac{7}{1+x^2}\,dx = 2\Big[7\arctan(x)\Big]_0^1 = 14\arctan(1) = 14\cdot\frac{\pi}{4} = \frac{7\pi}{2}$$

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Problem 5 Solution

5. A square piece of sheet metal measuring 12 inches by 12 inches is to form a box with a square base and no top by cutting the side length x from each corner, and then folding up the remaining sides. [See the picture below where the black squares represent the squares of side length x that are being removed, and the square with the dotted edge represents the base.]

(a) If $V(x)$ denotes the volume of the box when squares of side length x are removed from the corners, what is a formula representing $V(x)$?

(b) What are the possible values for x?

(c) What is the maximum possible volume of such a box? Make sure to use proper units.

Solution:

(a) The box has a square base whose sides have length $12 - 2x$. The height of the box is x. Therefore, the volume is

$$V(x) = (12 - 2x)^2 x = 144x - 48x^2 + 4x^3$$

(b) The interval is $0 \leq x \leq 6$.

(c) The critical point of f in the interval $(0, 6)$ is:

$$\begin{aligned} f'(x) &= 0 \\ 144 - 96x + 12x^2 &= 0 \\ 12(x^2 - 8x + 12) &= 0 \\ 12(x-2)(x-6) &= 0 \\ x &= 2 \end{aligned}$$

The volume is 0 at both $x = 0$ and $x = 6$. The volume at $x = 2$ is:

$$V(2) = (12 - 2 \cdot 2)^2 \cdot 2 = 128 \ \text{ in}^3$$

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Problem 6 Solution

6. Consider the function $f(x) = \dfrac{2x^2+5}{x^2-25}$.

   (a) Find the domain of f.

   (b) Find all vertical and horizontal asymptotes of f. [Justify your answers using calculus.]

   (c) Find the intervals where f is increasing and the intervals where f is decreasing.

   (d) Identify all local extrema of f.

Solution:

(a) The domain is $(-\infty,-5)\cup(-5,5)\cup(5,\infty)$.

(b) $x=\pm 5$ are vertical asymptotes because

$$\lim_{x\to 5^+}\frac{2x^2+5}{x^2-25}=\infty, \quad \lim_{x\to -5^+}\frac{2x^2+5}{x^2-25}=-\infty$$

$y=2$ is a horizontal asymptote because

$$\lim_{x\to\pm\infty}\frac{2x^2+5}{x^2-25}=2$$

(c) To determine where f is increasing and decreasing we must find all critical points of f. The derivative of f is:

$$f'(x)=\frac{(x^2-25)(4x)-(2x^2+5)(2x)}{(x^2-25)^2}=\frac{-110x}{(x^2-25)^2}$$

$f'(x)=0$ when $x=0$. Thus, we must evaluate $f'(x)$ at test points on the intervals $(-\infty,-5)$, $(-5,0)$, $(0,5)$, and $(5,\infty)$.

$$\begin{aligned} f'(-6) &= \frac{660}{121} > 0 \\ f'(-4) &= \frac{440}{81} > 0 \\ f'(4) &= \frac{-440}{81} < 0 \\ f'(6) &= \frac{-660}{121} < 0 \end{aligned}$$

Thus, f is increasing on $(-\infty,-5)\cup(-5,0)$ and is decreasing on $(0,5)\cup(5,\infty)$.

(d) Since f' changes sign from $+$ to $-$ across $x=0$, f attains a local maximum value at $x=0$.

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Problem 7 Solution

7. Consider the curve $e^{xy} + y = 2$.

   (a) Find $\dfrac{dy}{dx}$.

   (b) Write the equation of the tangent line to the curve at the point $(0, 1)$.

Solution:

(a) The derivative is computed using implicit differentiation.

$$
\begin{aligned}
\frac{d}{dx}e^{xy} + \frac{d}{dx}y &= \frac{d}{dx}2 \\
e^{xy}\frac{d}{dx}(xy) + \frac{dy}{dx} &= 0 \\
e^{xy}\left(x\frac{dy}{dx} + y\right) + \frac{dy}{dx} &= 0 \\
\frac{dy}{dx}\left(xe^{xy} + 1\right) &= -ye^{xy} \\
\frac{dy}{dx} &= -\frac{ye^{xy}}{xe^{xy} + 1}
\end{aligned}
$$

(b) The slope of the tangent line at $(0, 1)$ is:

$$\left.\frac{dy}{dx}\right|_{(0,1)} = -\frac{1 \cdot e^{0 \cdot 1}}{0 \cdot e^{0 \cdot 1} + 1} = -1$$

Thus, an equation for the tangent line in point-slope form is:

$$y = 1 = -(x - 0)$$

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Problem 8 Solution

8. Calculate $\displaystyle\int x\sqrt{x+4}\,dx$.

Solution: Let $u = x + 4$. Then $du = dx$ and $x = u - 4$. Making the substitutions yields

$$\int x\sqrt{x+4}\,dx = \int (u-4)\sqrt{u}\,du = \int \left(u^{3/2} - 4u^{1/2}\right)\,du = \frac{2}{5}u^{5/2} - \frac{8}{3}u^{3/2} + C = \frac{2}{5}(x+4)^{5/2} - \frac{8}{3}(x+4)^{3/2} + C$$