Math 180, Final Exam, Spring 2009
Problem 1 Solution

1. Differentiate the following functions. Do not simplify your answers.

(a) $x\tan^{-1}x$ (b) $2e^{\sqrt{x}}$ (c) $\dfrac{x^7+x}{\ln x}$ (d) $\sec(x^{3/2})$

Solution:

(a) Use the Product Rule.

$$\begin{aligned}(x\tan^{-1}x)' &= x(\tan^{-1}x)' + (x)'\tan^{-1}x \\ &= \boxed{x\cdot\frac{1}{1+x^2}+\tan^{-1}x}\end{aligned}$$

(b) Use the Chain Rule.

$$\begin{aligned}(2e^{\sqrt{x}})' &= 2e^{\sqrt{x}}\cdot(\sqrt{x})' \\ &= \boxed{2e^{\sqrt{x}}\cdot\left(\frac{1}{2\sqrt{x}}\right)}\end{aligned}$$

(c) Use the Quotient Rule.

$$\begin{aligned}\left(\frac{x^7+x}{\ln x}\right)' &= \frac{(\ln x)(x^7+x)'-(x^7+x)(\ln x)'}{(\ln x)^2} \\ &= \boxed{\frac{(\ln x)(7x^6+1)-(x^7+x)(\frac{1}{x})}{(\ln x)^2}}\end{aligned}$$

(d) Use the Chain Rule.

$$\begin{aligned}\left[\sec(x^{3/2})\right]' &= \sec(x^{3/2})\tan(x^{3/2})\cdot(x^{3/2})' \\ &= \boxed{\sec(x^{3/2})\tan(x^{3/2})\cdot\left(\frac{3}{2}x^{1/2}\right)}\end{aligned}$$

Math 180, Final Exam, Spring 2009
Problem 2 Solution

2. Let $f(x) = x^2 - 4x$. Find $f'(3)$ using the definition of the derivative as the limit of a difference quotient.

Solution: There are two possible difference quotients we can use to evaluate $f'(3)$. One is:

$$f'(3) = \lim_{h\to 0} \frac{f(h+3) - f(3)}{h} = \lim_{h\to 0} \frac{[(h+3)^2 - 4(h+3)] - [3^2 - 4(3)]}{h}.$$

The other is:

$$f'(3) = \lim_{x\to 3} \frac{f(x) - f(3)}{x-3} = \lim_{x\to 3} \frac{(x^2 - 4x) - [3^2 - 4(3)]}{x-3}$$

Evaluating the first limit above we have:

$$\begin{aligned}
f'(3) &= \lim_{h\to 0} \frac{[(h+3)^2 - 4(h+3)] - [3^2 - 4(3)]}{h} \\
&= \lim_{h\to 0} \frac{h^2 + 6h + 9 - 4h - 12 - (-3)}{h} \\
&= \lim_{h\to 0} \frac{h^2 + 2h}{h} \\
&= \lim_{h\to 0} (h+2) \\
&= 0 + 2 \\
&= \boxed{2}
\end{aligned}$$

Math 180, Final Exam, Spring 2009
Problem 3 Solution

3. Consider the function:

$$f(x) = \begin{cases} \sin(\pi x) & \text{for } x \le 1 \\ x^2 - 2x + 1 & \text{for } x > 1 \end{cases}$$

(a) Find the one-sided limits: $\lim\limits_{x\to 1^-} f(x)$, $\lim\limits_{x\to 1^+} f(x)$.

(b) Does $\lim\limits_{x\to 1} f(x)$ exist? Why or why not?

(c) Is $f(x)$ continuous at $x = 1$? Why or why not?

Solution:

(a) The one-sided limits are calculated as follows:

$$\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} \sin(\pi x) \\ &= \sin(\pi \cdot 1) \\ &= \boxed{0} \\ \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} (x^2 - 2x + 1) \\ &= 1^1 - 2(1) + 1 \\ &= \boxed{0} \end{aligned}$$

(b) The limit $\lim\limits_{x\to 1} f(x)$ exists because the one-sided limits exist and are equal to each other. In fact, $\lim\limits_{x\to 1} f(x) = 0$.

(c) The value of $f(x)$ at $x = 1$ is $f(1) = \sin(\pi \cdot 1) = 0$. Thus, since $f(1) = \lim\limits_{x\to 1} f(x) = 0$ we know that $f(x)$ is continuous at $x = 1$.

Math 180, Final Exam, Spring 2009
Problem 4 Solution

4. Consider the curve in the xy-plane defined by the equation $-xy + \ln y = 1$.

(a) Find an expression for $\frac{dy}{dx}$, the derivative of y with respect to x, in terms of x and y.

(b) Find an equation of the line tangent to the curve at $(-1, 1)$.

Solution:

(a) We must find $\frac{dy}{dx}$ using implicit differentiation.

$$\begin{aligned}
-xy + \ln y &= 1 \\
\frac{d}{dx}(-xy) + \frac{d}{dx}\ln y &= \frac{d}{dx}1 \\
\left(-x\frac{dy}{dx} - y\right) + \frac{1}{y}\cdot\frac{dy}{dx} &= 0 \\
-x\frac{dy}{dx} + \frac{1}{y}\cdot\frac{dy}{dx} &= y \\
\frac{dy}{dx}\left(-x + \frac{1}{y}\right) &= y \\
\frac{dy}{dx} &= \frac{y}{-x + \frac{1}{y}} \\
\frac{dy}{dx} &= \boxed{\frac{y^2}{-xy + 1}}
\end{aligned}$$

(b) The value of $\frac{dy}{dx}$ at $(-1, 1)$ is the slope of the tangent line.

$$\left.\frac{dy}{dx}\right|_{(-1,1)} = \frac{1^2}{-(-1)(1) + 1} = \frac{1}{2}$$

An equation for the tangent line at $(-1, 1)$ is then:

$$\boxed{y - 1 = \frac{1}{2}(x + 1)}$$

Math 180, Final Exam, Spring 2009
Problem 5 Solution

5. Compute each limit below.

(a) $\lim_{x\to 4} \dfrac{x-4}{\sqrt{x}-2}$

(b) $\lim_{x\to 0} \dfrac{e^x - \sin x - 1}{x^2 - x^3}$

Solution:

(a) Upon substituting $x = 4$ into the function $f(x) = \dfrac{x-4}{\sqrt{x}-2}$ we find that

$$\frac{x-4}{\sqrt{x}-2} = \frac{4-4}{\sqrt{4}-2} = \frac{0}{0}$$

which is indeterminate. We can resolve the indeterminacy by multiplying $f(x)$ by the "conjugate" of the denominator divided by itself.

$$\begin{aligned}
\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2} &= \lim_{x\to 4} \frac{x-4}{\sqrt{x}-2} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} \\
&= \lim_{x\to 4} \frac{(x-4)(\sqrt{x}+2)}{x-4} \\
&= \lim_{x\to 4} (\sqrt{x}+2) \\
&= \sqrt{4}+2 \\
&= \boxed{4}
\end{aligned}$$

(b) When substituting $x = 0$ into the function $f(x) = \dfrac{e^x - \sin x - 1}{x^2 - x^3}$ we find that

$$\frac{e^x - \sin x - 1}{x^2 - x^3} = \frac{e^0 - \sin 0 - 1}{0^2 - 0^3} = \frac{0}{0}$$

which is indeterminate. We resolve the indeterminacy by using L'Hôpital's Rule.

$$\begin{aligned}
\lim_{x\to 0} \frac{e^x - \sin x - 1}{x^2 - x^3} &\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{(e^x - \sin x - 1)'}{(x^2 - x^3)'} \\
&= \lim_{x\to 0} \frac{e^x - \cos x}{2x - 3x^2}
\end{aligned}$$

Upon substituting $x = 0$ we obtain the indeterminate form $\frac{0}{0}$ again. Thus, we apply L'Hôpital's Rule one more time.

$$\begin{aligned}
\lim_{x\to 0} \frac{e^x - \cos x}{2x - 3x^2} &\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{(e^x - \cos x)'}{(2x - 3x^2)'} \\
&= \lim_{x\to 0} \frac{e^x + \sin x}{2 - 6x} \\
&= \frac{e^0 + \sin 0}{2 - 6(0)} \\
&= \boxed{\frac{1}{2}}
\end{aligned}$$

Math 180, Final Exam, Spring 2009
Problem 6 Solution

6. A 300 square foot rectangular garden is to be surrounded on three sides by shrubbery costing \$10 per foot and on the remaining side by a fence costing \$5 per foot. Find the dimensions (and length of fence) of the least costly solution.

Solution: We begin by letting x be the length of the fence and the length of the side with shrubs opposite the wall and letting y be the lengths of the remaining two sides with shrubs. The function we seek to minimize is the cost:

$$\textbf{Function}: \qquad \text{Cost} = \$5x + \$10x + \$10y + \$10y \tag{1}$$

where $\$5x$ is the cost of making the wall and the remaining terms are the costs of the shrubs. The constraint in this problem is that the area of the garden is 300 square feet.

$$\textbf{Constraint}: \qquad xy = 300 \tag{2}$$

Solving the constraint equation (2) for y we get:

$$y = \frac{300}{x} \tag{3}$$

Plugging this into the function (1) and simplifying we get:

$$\begin{aligned} \text{Cost} &= \$5x + \$10x + \$10\left(\frac{300}{x}\right) + \$10\left(\frac{300}{x}\right) \\ f(x) &= 15x + \frac{6000}{x} \end{aligned}$$

We want to find the absolute minimum of $f(x)$ on the **interval** $(0, \infty)$. We choose this interval because x must be nonnegative (x represents a length) and non-zero (if x were 0, then the area would be 0 but it must be 300).

The absolute minimum of $f(x)$ will occur either at a critical point of $f(x)$ in $(0, \infty)$ or it will not exist because the interval is open. The critical points of $f(x)$ are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left(15x + \frac{6000}{x}\right)' &= 0 \\ 15 - \frac{6000}{x^2} &= 0 \\ 15x^2 - 6000 &= 0 \\ x^2 &= 400 \\ x &= \pm 20 \end{aligned}$$

However, since $x = -20$ is outside $(0, \infty)$, the only critical point is $x = 20$. Plugging this into $f(x)$ we get:

$$f(20) = 15(20) + \frac{6000}{20} = 600$$

Taking the limits of $f(x)$ as x approaches the endpoints we get:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \left(15x + \frac{6000}{x}\right) = 0 + \infty = \infty$$
$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty} \left(15x + \frac{6000}{x}\right) = \infty + 0 = \infty$$

both of which are larger than 600. We conclude that the cost is an absolute minimum at $\boxed{x = 20}$ and that the resulting cost is \$600. The last step is to find the corresponding value for y by plugging $x = 20$ into equation (3).

$$y = \frac{300}{x}$$
$$y = \frac{300}{20}$$
$$\boxed{y = 15}$$

Math 180, Final Exam, Spring 2009
Problem 7 Solution

7.

(a) Use the Fundamental Theorem of Calculus to compute:

$$\text{i. } \int_1^4 (x^2 - x^{-3/2})\,dx \qquad \text{ii. } \frac{d}{dx}\int_1^x \frac{dt}{1+t^2}$$

(b) Check your answer to ii. by first evaluating the integral and then differentiating your result.

Solution:

(a) i. Using FTC I, we have:

$$\begin{aligned}\int_1^4 \left(x^2 - x^{-3/2}\right)\,dx &= \frac{x^3}{3} + \frac{2}{x^{1/2}}\bigg|_1^4 \\ &= \left(\frac{4^3}{3} + \frac{2}{\sqrt{4}}\right) - \left(\frac{1^3}{3} + \frac{2}{\sqrt{1}}\right) \\ &= \boxed{20}\end{aligned}$$

ii. Using FTC II, we have:

$$\frac{d}{dx}\int_1^x \frac{dt}{1+t^2} = \boxed{\frac{1}{1+x^2}}$$

(b) Checking our answer to ii. above we get:

$$\begin{aligned}\frac{d}{dx}\int_1^x \frac{dt}{1+t^2} &= \frac{d}{dx}\Big[\arctan t\Big]_1^x \\ &= \frac{d}{dx}\Big[\arctan x - \arctan 1\Big] \\ &= \frac{d}{dx}\arctan x - \frac{d}{dx}\arctan 1 \\ &= \frac{1}{1+x^2} - 0 \\ &= \boxed{\frac{1}{1+x^2}}\end{aligned}$$

Math 180, Exam 2, Spring 2009
Problem 8 Solution

8. Consider the function $f(x) = x^3 - 6x$.

(a) Find the critical points of f and classify them as being a local minimum, a local maximum, or neither.

(b) On what interval(s) is f decreasing?

(c) On what interval(s) is f concave up?

(d) Find the absolute minimum and maximum of f on the interval $[-2, 1]$.

Solution:

(a) The critical points of $f(x)$ are the values of x for which either $f'(x) = 0$ or $f'(x)$ does not exist. Since $f(x)$ is a polynomial, $f'(x)$ exists for all $x \in \mathbb{R}$. Therefore, the only critical points are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left(x^3 - 6x\right)' &= 0 \\ 3x^2 - 6 &= 0 \\ 3(x^2 - 2) &= 0 \\ x^2 &= 2 \\ x &= \pm\sqrt{2} \end{aligned}$$

We use the Second Derivative Test to classify the critical points $\boxed{x = -\sqrt{2}}$ and $\boxed{x = \sqrt{2}}$. The second derivative is $f''(x) = 6x$. At the critical points, we have:

$$\begin{aligned} f''(-\sqrt{2}) &= 6(-\sqrt{2}) = -6\sqrt{2} \\ f''(\sqrt{2}) &= 6(\sqrt{2}) = 6\sqrt{2} \end{aligned}$$

Since $f''(-\sqrt{2}) < 0$ the Second Derivative Test implies that $f(-\sqrt{2}) = 4\sqrt{2}$ is a local maximum. Since $f''(\sqrt{2}) > 0$ the Second Derivative Test implies that $f(\sqrt{2}) = -4\sqrt{2}$ is a local minimum.

(b) A function $f(x)$ is decreasing on (a, b) when $f'(x) < 0$ for all $x \in (a, b)$. To find the interval(s) where f is decreasing, we split the domain of $f(x)$, which is $(-\infty, \infty)$, into the intervals $(-\infty, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$, and $(\sqrt{2}, \infty)$ and evaluate $f'(x)$ at test points in each interval.

Interval	**Test Number,** c	$f'(c)$	**Sign of** $f'(c)$
$(-\infty, -\sqrt{2})$	-2	6	$+$
$(-\sqrt{2}, \sqrt{2})$	0	-6	$-$
$(\sqrt{2}, \infty)$	2	6	$+$

Since $f'(0) = -6 < 0$, we know that f is decreasing on the interval $\boxed{(-\sqrt{2}, \sqrt{2})}$.

(c) A function $f(x)$ is concave up on (a, b) when $f''(x) > 0$ for all $x \in (a, b)$. To find the interval(s) where f is concave up, we must first determine the value(s) of x for which $f''(x) = 0$.

$$\begin{aligned} f''(x) &= 0 \\ \left(3x^2 - 6\right)' &= 0 \\ 6x &= 0 \\ x &= 0 \end{aligned}$$

Since the domain of f is $(-\infty, \infty)$, we divide the domain into the two intervals $(-\infty, 0)$ and $(0, \infty)$. We now evaluate f'' at test points in each interval to determine where $f''(x)$ is positive and negative.

Interval	**Test Number,** c	$f''(c)$	**Sign of** $f''(c)$
$(-\infty, 0)$	-1	-6	$-$
$(0, \infty)$	1	6	$+$

Since $f''(1) = 6 > 0$, we know that f is concave up on the interval $\boxed{(0, \infty)}$.

(d) The absolute extrema of f will occur either at a critical point in $[-2, 1]$ or at one of the endpoints. From part (a), we found that the critical points of f are $x = -\sqrt{2}$ and $x = \sqrt{2}$. The point $x = \sqrt{2}$ is outside the interval, though, so we only evaluate f at $x = -2$, $x = -\sqrt{2}$, and $x = 1$.

$$\begin{aligned} f(-2) &= (-2)^3 - 6(-2) = 4 \\ f(-\sqrt{2}) &= (-\sqrt{2})^3 - 6(-\sqrt{2}) = 4\sqrt{2} \\ f(1) &= 1^3 - 6(1) = -5 \end{aligned}$$

The absolute minimum of f on $[-2, 1]$ is $\boxed{-5}$ because it is the smallest of the values of f above and the absolute maximum is $\boxed{4\sqrt{2}}$ because it is the largest.

Math 180, Final Exam, Spring 2009
Problem 9 Solution

9. Use Calculus to compute the area of the region bounded by the vertical lines $x = 1$ and $x = 4$, by the horizontal line $y = 0$, and by the curve $y = x^2 + 2\sqrt{x}$.

Solution: The area of the region is:

$$\begin{aligned}
\text{Area} &= \int_1^4 \left(x^2 + 2\sqrt{x}\right)\, dx \\
&= \frac{x^3}{3} + \frac{4}{3}x^{3/2}\Big|_1^4 \\
&= \left(\frac{4^3}{3} + \frac{4}{3}(4)^{3/2}\right) - \left(\frac{1^3}{3} + \frac{4}{3}(1)^{3/2}\right) \\
&= \boxed{\frac{91}{3}}
\end{aligned}$$

Math 180, Final Exam, Spring 2009
Problem 10 Solution

10. Evaluate each of the following indefinite integrals:

(a) $\displaystyle\int [\sin(2x+1) + e^{-x}]\,dx$ (b) $\displaystyle\int \frac{dx}{x(\ln x)^2}$

Solution:

(a) $\displaystyle\int [\sin(2x+1) + e^{-x}]\,dx = \boxed{-\frac{1}{2}\cos(2x+1) - e^{-x} + C}$

(b) To evaluate this integral, we use the substitution $u = \ln x$, $du = \dfrac{1}{x}\,dx$. The integral then becomes:

$$\begin{aligned}\int \frac{dx}{x(\ln x)^2} &= \int \frac{du}{u^2} \\ &= -\frac{1}{u} + C \\ &= \boxed{-\frac{1}{\ln x} + C}\end{aligned}$$