Math 181, Exam 1, Fall 2007
Problem 1 Solution

1. A function f satisfies $f''(x) = 3x - \sin x$, $f'(0) = 1$, and $f(0) = 2$. Find f.

Solution: The function $f'(x)$ is obtained by integrating $f''(x)$:

$$\begin{aligned} f'(x) &= \int f''(x)\,dx \\ f'(x) &= \int (3x - \sin x)\,dx \\ f'(x) &= \frac{3}{2}x^2 + \cos x + C_1 \end{aligned}$$

The value of C_1 is found by using the fact that $f'(0) = 1$.

$$\begin{aligned} f'(0) &= 1 \\ \frac{3}{2}(0)^2 + \cos 0 + C_1 &= 1 \\ C_1 &= 0 \end{aligned}$$

The function $f(x)$ is obtained by integrating $f'(x)$:

$$\begin{aligned} f(x) &= \int f'(x)\,dx \\ f(x) &= \int \left(\frac{3}{2}x^2 + \cos x\right)\,dx \\ f(x) &= \frac{1}{2}x^3 + \sin x + C_2 \end{aligned}$$

The value of C_2 is found by using the fact that $f(0) = 2$.

$$\begin{aligned} f(0) &= 2 \\ \frac{1}{2}(0)^3 + \sin 0 + C_2 &= 2 \\ C_2 &= 2 \end{aligned}$$

Therefore, the function $f(x)$ is:

$$\boxed{f(x) = \frac{1}{2}x^3 + \sin x + 2}$$

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Problem 2 Solution

2. Differentiate the function:

$$F(x) = \int_{\cos x}^{\ln x} \frac{dt}{\cos t + 2}$$

Solution: Using the Fundamental Theorem of Calculus Part II and the Chain Rule, the derivative of $F(x) = \displaystyle\int_{g(x)}^{h(x)} f(t)\,dt$ is:

$$\begin{aligned} F'(x) &= \frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt \\ &= f(h(x)) \cdot \frac{d}{dx}h(x) - f(g(x)) \cdot \frac{d}{dx}g(x) \end{aligned}$$

Applying the formula to the given function $F(x)$ we get:

$$\begin{aligned} F'(x) &= \frac{d}{dx}\int_{\cos x}^{\ln x} \frac{dt}{\cos t + 2} \\ &= \frac{1}{\cos(\ln x) + 2} \cdot \frac{d}{dx}(\ln x) - \frac{1}{\cos(\cos x) + 2} \cdot \frac{d}{dx}(\cos x) \\ &= \boxed{\frac{1}{\cos(\ln x) + 2} \cdot \left(\frac{1}{x}\right) - \frac{1}{\cos(\cos x) + 2} \cdot (-\sin x)} \end{aligned}$$

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Problem 3 Solution

3. Compute the definite integral:

$$\int_{-\pi}^{\pi} x\cos(2x)\,dx$$

Solution: We will evaluate the integral using Integration by Parts. Let $u = x$ and $v' = \cos(2x)$. Then $u' = 1$ and $v = \frac{1}{2}\sin(2x)$. Using the Integration by Parts formula:

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$

we get:

$$\begin{aligned}
\int_{-\pi}^{\pi} x\cos(2x)\,dx &= \left[\frac{1}{2}x\sin(2x)\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{1}{2}\sin(2x)\,dx \\
&= \left[\frac{1}{2}x\sin(2x)\right]_{-\pi}^{\pi} - \left[-\frac{1}{4}\cos(2x)\right]_{-\pi}^{\pi} \\
&= \left[\frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x)\right]_{-\pi}^{\pi} \\
&= \left[\frac{1}{2}\pi\sin(2\pi) + \frac{1}{4}\cos(2\pi)\right] - \left[-\frac{1}{2}\pi\sin(-2\pi) + \frac{1}{4}\cos(-2\pi)\right] \\
&= \boxed{0}
\end{aligned}$$

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Problem 4 Solution

4. Compute the indefinite integrals:

$$\int (\ln x)^2\,dx \qquad \int \frac{3x-4}{x^2-3x+2}\,dx \qquad \int \frac{(\ln x)^3}{x}\,dx \qquad \int x\sqrt{3x+1}\,dx$$

Solution:

- The first integral is computed using Integration by Parts. Let $u = (\ln x)^2$ and $v' = 1$. Then $u' = \dfrac{2\ln x}{x}$ and $v = x$. Using the Integration by Parts formula:

  $$\int uv'\,dx = uv - \int u'v\,dx$$

  we get:

  $$\begin{aligned}\int (\ln x)^2\,dx &= x(\ln x)^2 - \int \left(\frac{2\ln x}{x}\right)(x)\,dx \\ &= x(\ln x)^2 - 2\int \ln x\,dx\end{aligned}$$

  A second Integration by Parts must be performed. Let $u = \ln x$ and $v' = 1$. Then $u' = \dfrac{1}{x}$ and $v = x$. Using the Integration by Parts formula again we get:

  $$\begin{aligned}\int (\ln x)^2\,dx &= x(\ln x)^2 - 2\int \ln x\,dx \\ &= x(\ln x)^2 - 2\left(x\ln x - \int \left(\frac{1}{x}\right)x\,dx\right) \\ &= x(\ln x)^2 - 2\left(x\ln x - \int dx\right) \\ &= \boxed{x(\ln x)^2 - 2x\ln x + 2x + C}\end{aligned}$$

- The second integral is computed using Partial Fraction Decomposition. Factoring the denominator and decomposing we get:

  $$\frac{3x-4}{x^2-3x+2} = \frac{3x-4}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

  Multiplying the equation by $(x-1)(x-2)$ we get:

  $$3x - 4 = A(x-2) + B(x-1)$$

Next we plug in two different values of x to get a system of two equations in two unknowns (A, B). Letting $x = 1$ and $x = 2$ we get:

$$\begin{aligned} x = 1: &\quad 3(1) - 4 = A(1-2) + B(1-1) \quad \Rightarrow \quad A = 1 \\ x = 2: &\quad 3(2) - 4 = A(2-2) + B(2-1) \quad \Rightarrow \quad B = 2 \end{aligned}$$

Plugging these values of A and B back into the decomposed equation and integrating we get:

$$\begin{aligned} \int \frac{3x-4}{x^2-3x+2}\,dx &= \int \left(\frac{1}{x-1} + \frac{2}{x-2} \right) dx \\ &= \boxed{\ln|x-1| + 2\ln|x-2| + C} \end{aligned}$$

- The third integral is computed using the u-substitution method. Let $u = \ln x$. Then $du = \dfrac{1}{x}\,dx$. We get:

$$\begin{aligned} \int \frac{(\ln x)^3}{x}\,dx &= \int u^3\,du \\ &= \frac{1}{4}u^4 + C \\ &= \boxed{\frac{1}{4}(\ln x)^4 + C} \end{aligned}$$

- The fourth integral is computed using the u-substitution. Let $u = 3x+1$. Then $du = 3\,dx \quad \Rightarrow \quad \dfrac{1}{3}\,du = dx$ and $x = \dfrac{1}{3}(u-1)$ and we get:

$$\begin{aligned} \int x\sqrt{3x+1}\,dx &= \int \frac{1}{3}(u-1)\sqrt{u}\left(\frac{1}{3}\,du\right) \\ &= \frac{1}{9}\int \left(u^{3/2} - u^{1/2}\right) du \\ &= \frac{1}{9}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C \\ &= \boxed{\frac{2}{45}(3x+1)^{5/2} - \frac{2}{27}(3x+1)^{3/2} + C} \end{aligned}$$