Math 181, Exam 1, Fall 2013
Problem 1 Solution

1. Compute the integrals

   (a) $\displaystyle\int \sin^{-1}(x)\,dx$

   (b) $\displaystyle\int \frac{dx}{x^2(x+1)}$

   (c) $\displaystyle\int_0^3 \sqrt{9-x^2}\,dx$

Solution:

(a) Use Integration by Parts to evaluate the integral. Letting $u = \sin^{-1}(x)$ and $dv = dx$ yields

$$du = \frac{1}{\sqrt{1-x^2}}\,dx, \quad v = x.$$

Then we have

$$\int u\,dv = uv - \int v\,du$$
$$\int \sin^{-1}(x)\,dx = x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx.$$

To evaluate the integral on the right hand side of the above equation, we let $u = 1-x^2$ and $du = -2x\,dx$ so $-\frac{1}{2}\,du = x\,dx$. Making these substitutions we obtain:

$$\int \sin^{-1}(x)\,dx = x\sin^{-1}(x) + \int \frac{1}{2\sqrt{u}}\,du$$
$$\int \sin^{-1}(x)\,dx = x\sin^{-1}(x) + \sqrt{u} + C$$
$$\boxed{\int \sin^{-1}(x)\,dx = x\sin^{-1}(x) + \sqrt{1-x^2} + C}$$

(b) Use the method of partial fractions. The decomposition of the integrand is

$$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}.$$

After clearing denominators we obtain

$$1 = Ax(x+1) + B(x+1) + Cx^2.$$

Letting $x = 0$ yields $B = 1$ and letting $x = -1$ yields $C = 1$. After expanding the right hand side of the above equation we obtain

$$1 = x^2(A + C) + x(A + B) + B.$$

Equating the coefficient of x^2 on both sides of the equation yields $0 = A + C$ so $A = -C = -1$. Thus, the decomposition is

$$\frac{1}{x^2(x+1)} = -\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}.$$

The integral is then

$$\int \frac{dx}{x^2(x+1)} = \int \left(-\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right) dx$$

$$\boxed{\int \frac{dx}{x^2(x+1)} = -\ln|x| - \frac{1}{x} + \ln|x+1| + C}$$

(c) The integral represents one-fourth of the area of a circle of radius 3. That is,

$$\boxed{\int_0^3 \sqrt{9 - x^2}\, dx = \frac{1}{4}\pi(3)^2 = \frac{9\pi}{4}}.$$

The other method of solution is to use the trigonometric substitution

$$x = 3\sin\theta, \quad dx = 3\cos\theta\, d\theta.$$

When $x = 0$ we have $\sin\theta = 0$ and, thus, $\theta = 0$. When $x = 3$ we have $\sin\theta = 1$ and, thus, $\theta = \frac{\pi}{2}$. The definite integral is then converted and evaluated as follows:

$$\int_0^3 \sqrt{9 - x^2}\, dx = \int_0^{\pi/2} \sqrt{9 - (3\sin\theta)^2} \cdot 3\cos\theta\, d\theta$$

$$\int_0^3 \sqrt{9 - x^2}\, dx = \int_0^{\pi/2} \sqrt{9 - 9\sin^2\theta} \cdot 3\cos\theta\, d\theta$$

$$\int_0^3 \sqrt{9 - x^2}\, dx = \int_0^{\pi/2} \sqrt{9(1 - \sin^2\theta)} \cdot 3\cos\theta\, d\theta$$

$$\int_0^3 \sqrt{9 - x^2}\, dx = \int_0^{\pi/2} \sqrt{9\cos^2\theta} \cdot 3\cos\theta\, d\theta$$

$$\int_0^3 \sqrt{9 - x^2}\, dx = \int_0^{\pi/2} 3\cos\theta \cdot 3\cos\theta\, d\theta$$

$$\int_0^3 \sqrt{9 - x^2}\, dx = \int_0^{\pi/2} 9\cos^2\theta\, d\theta$$

$$\int_0^3 \sqrt{9 - x^2}\, dx = 9\left[\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right]_0^{\pi/2}$$

$$\boxed{\int_0^3 \sqrt{9 - x^2}\, dx = \frac{9\pi}{4}}$$

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Problem 2 Solution

2. Compute the length of the graph of $f(x) = \dfrac{e^x + e^{-x}}{2}$ from $x = 0$ to $x = \ln(2)$.

Solution: The arclength formula is

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

where

$$f'(x) = \frac{e^x - e^{-x}}{2}.$$

The quantity $1 + f'(x)^2$ simplifies as follows:

$$\begin{aligned}
1 + f'(x)^2 &= 1 + \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
1 + f'(x)^2 &= 1 + \frac{(e^x - e^{-x})^2}{4} \\
1 + f'(x)^2 &= 1 + \frac{e^{2x} - 2 + e^{-2x}}{4} \\
1 + f'(x)^2 &= \frac{4 + e^{2x} - 2 + e^{-2x}}{4} \\
1 + f'(x)^2 &= \frac{e^{2x} + 2 + e^{-2x}}{4} \\
1 + f'(x)^2 &= \left(\frac{e^x + e^{-x}}{2}\right)^2
\end{aligned}$$

Therefore, the arclength is

$$\begin{aligned}
L &= \int_0^{\ln(2)} \sqrt{\left(\frac{e^x + e^{-x}}{2}\right)^2}\, dx \\
L &= \int_0^{\ln(2)} \frac{e^x + e^{-x}}{2}\, dx \\
L &= \left.\frac{e^x - e^{-x}}{2}\right|_0^{\ln(2)} \\
L &= \frac{e^{\ln(2)} - e^{-\ln(2)}}{2} \\
L &= \frac{2 - \frac{1}{2}}{2} \\
&\boxed{L = \frac{3}{4}}
\end{aligned}$$

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Problem 3 Solution

3. Consider the region enclosed by $y = 5 - x^2$ the y-axis and $y = 1$. Find the volume of revolution of the resulting solid, when the region is rotated about:

(a) the x-axis,

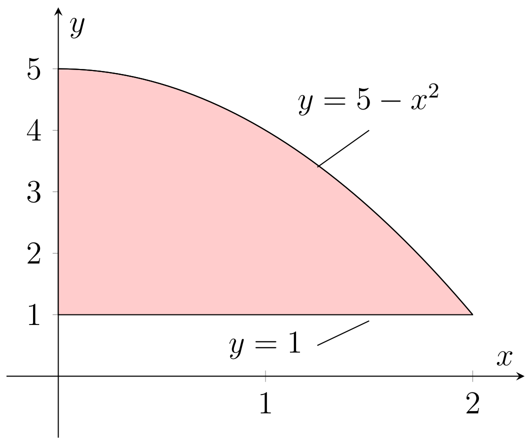
(b) the axis $x = -2$.

Solution:

(a) The volume is obtained using the Washer Method. The corresponding formula is

$$V = \pi \int_a^b \pi[f(x)^2 - g(x)^2]\, dx.$$

A sketch of the region enclosed by the given curves is shown below.



From the sketch of the region, we know that $f(x) = 5 - x^2$ and $g(x) = 1$. Thus, the volume is

$$V = \pi \int_0^2 [(5 - x^2)^2 - 1^2]\, dx$$

$$V = \pi \int_0^2 \left(25 - 10x^2 + x^4 - 1\right)\, dx$$

$$V = \pi \int_0^2 \left(x^4 - 10x^2 + 24\right)\, dx$$

$$V = \pi \left[\frac{1}{5}x^5 - \frac{10}{3}x^3 + 24x\right]_0^2$$

$$V = \pi \left[\frac{32}{5} - \frac{80}{3} + 48\right]$$

$$\boxed{V = \frac{416\pi}{15}}$$

(b) Upon rotating about the axis $x = -2$, we use the Shell Method to find the corresponding volume. The formula we use is

$$V = 2\pi \int_a^b (x+2)[f(x) - g(x)]\,dx$$

where the shell radius is $x + 2$. Using the definitions of $f(x)$ and $g(x)$ from part (a) we have

$$V = 2\pi \int_0^2 (x+2)(5 - x^2 - 1)\,dx$$

$$V = 2\pi \int_0^2 (x+2)(4 - x^2)\,dx$$

$$V = 2\pi \int_0^2 (4x - x^3 + 8 - 2x^2)\,dx$$

$$V = 2\pi \left[2x^2 - \frac{1}{4}x^4 + 8x - \frac{2}{3}x^3\right]_0^2$$

$$V = 2\pi \left[2(4) - \frac{1}{4}(4) + 8(4) - \frac{2}{3}(2)^3\right]$$

$$\boxed{V = \frac{88\pi}{3}}$$

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Problem 4 Solution

4. Compute the area of each region below.

(a) the region between $y = x\sqrt{4-x}$ and the x-axis from $x = 0$ to $x = 3$

(b) the region between the graphs of $y = 5 - x^2$ and $y = 3 - x$

Solution:

(a) The area of the region is

$$A = \int_0^3 x\sqrt{4-x}\,dx.$$

Using the substitution $u = 4 - x$ we obtain $-du = dx$ and $x = 4 - u$. The limits of integration become:

- $x = 0 \Rightarrow u = 4 - 0 = 4$
- $x = 3 \Rightarrow u = 4 - 3 = 1$

Thus, the area is

$$A = -\int_4^1 (4-u)\sqrt{u}\,du$$
$$A = \int_1^4 \left(4u^{1/2} - u^{3/2}\right)\,du$$
$$A = \left[\frac{8}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_1^4$$
$$A = \left[\frac{8}{3}(4)^{3/2} - \frac{2}{5}(4)^{5/2}\right] - \left[\frac{8}{3}(1)^{3/2} - \frac{2}{5}(1)^{5/2}\right]$$
$$A = \frac{64}{3} - \frac{64}{5} - \frac{8}{3} + \frac{2}{5}$$
$$\boxed{A = \frac{94}{15}}$$

(b) The graphs intersect when $y = y$. That is,

$$3 - x = 5 - x^2$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$
$$x = 2,\ x = -1$$

The graph of $y = 5 - x^2$ is above the graph of $y = 3 - x$ on the interval $-1 \leq x \leq 2$. Therefore, the area is

$$A = \int_{-1}^{2} \left[(5 - x^2) - (3 - x)\right] \, dx$$

$$A = \int_{-1}^{2} \left(2 + x - x^2\right) \, dx$$

$$A = \left[2x + \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-1}^{2}$$

$$A = \left[2(2) + \frac{1}{2}(2)^2 - \frac{1}{3}(2)^3\right] - \left[2(-1) + \frac{1}{2}(-1)^2 - \frac{1}{3}(-1)^3\right]$$

$$A = 4 + 2 - \frac{8}{3} + 2 - \frac{1}{2} - \frac{1}{3}$$

$$\boxed{A = \frac{9}{2}}$$

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Problem 5 Solution

5. Evaluate the indefinite integral

$$\int \frac{dx}{e^{2x} + e^x}.$$

Consider using the substitution $u = e^x$.

Solution: Letting $u = e^x$ yields $du = e^x\,dx$. In other words, $\frac{du}{u} = dx$ since $u = e^x$. The integral converts as follows:

$$\int \frac{dx}{e^{2x} + e^x} = \int \frac{du/u}{u^2 + u} = \int \frac{du}{u(u^2 + u)} = \int \frac{du}{u^2(u + 1)}$$

This integral was solved in Problem 1(b). The answer is

$$\int \frac{du}{u^2(u+1)} = -\ln|u| - \frac{1}{u} + \ln|u+1| + C$$

Using the fact that $u = e^x$ yields

$$\boxed{\int \frac{dx}{e^{2x} + e^x} = -\ln|e^x| - \frac{1}{e^x} + \ln|e^x + 1| + C}$$