Math 181, Exam 1, Study Guide 2
Problem 1 Solution

1. Compute the definite integral:

$$\int_1^5 \left(\frac{17}{x} + 3\right)\,dx$$

Solution: Using the Fundamental Theorem of Calculus Part I, the value of the integral is:

$$\begin{aligned}
\int_1^5 \left(\frac{17}{x} + 3\right)\,dx &= \Big[17\ln|x| + 3x\Big]_1^5 \\
&= [17\ln|5| + 3(5)] - [17\ln|1| + 3(1)] \\
&= 17\ln 5 + 15 - 0 - 3 \\
&= \boxed{17\ln 5 + 12}
\end{aligned}$$

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Problem 2 Solution

2. Consider the function $f(x) = 2x - x^2$ on the interval $[0, 2]$. Compute the trapezoid and midpoint approximations T_2 and M_2.

Solution: The length of each subinterval of $[0, 2]$ is:

$$\Delta x = \frac{b-a}{N} = \frac{2-0}{2} = 1$$

The trapezoid approximation T_2 is:

$$\begin{aligned} T_2 &= \frac{\Delta x}{2}\left[f(0) + 2f(1) + f(2)\right] \\ &= \frac{1}{2}\left[(2\cdot 0 - 0^2) + 2(2\cdot 1 - 1^2) + (2\cdot 2 - 2^2)\right] \\ &= \frac{1}{2}\left[0 + 2 + 0\right] \\ &= \boxed{1} \end{aligned}$$

The midpoint approximation M_2 is:

$$\begin{aligned} M_2 &= \Delta x\left[f\left(\frac{1}{2}\right) + f\left(\frac{3}{2}\right)\right] \\ &= 1\cdot\left[\left(2\cdot\frac{1}{2} - \left(\frac{1}{2}\right)^2\right) + \left(2\cdot\frac{3}{2} - \left(\frac{3}{2}\right)^2\right)\right] \\ &= \frac{3}{4} + \frac{3}{4} \\ &= \boxed{\frac{3}{2}} \end{aligned}$$

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Problem 3 Solution

3. The region enclosed by the graphs of the functions $y = x$ and $y = \sqrt{x}$ from $x = 0$ to $x = 1$ is rotated about the y-axis. Compute the volume of the resulting solid.

Solution: We will use the Shell Method to compute the volume. The formula is:

$$V = 2\pi \int_a^b x\,(\text{top} - \text{bottom})\; dx$$

where the top curve is $y = \sqrt{x}$, the bottom curve is $y = x$, the interval is $0 \leq x \leq 1$. Therefore, the volume is:

$$\begin{aligned} V &= 2\pi \int_0^1 x\left(\sqrt{x} - x\right)\; dx \\ &= 2\pi \int_0^1 \left(x^{3/2} - x^2\right)\; dx \\ &= 2\pi \left[\frac{2}{5}x^{5/2} - \frac{1}{3}x^3\right]_0^1 \\ &= 2\pi \left[\frac{2}{5} - \frac{1}{3}\right] \\ &= \boxed{\frac{2\pi}{15}} \end{aligned}$$

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Problem 4 Solution

4. Compute the following integrals:

$$\int \sin^2 x \cos^3 x \, dx \qquad \int \frac{1}{\sqrt{4-x^2}} \, dx$$

Solution: The first integral is computed by rewriting the integral using the Pythagorean Identity $\cos^2 x + \sin^2 x = 1$.

$$\begin{aligned}
\int \sin^2 x \cos^3 x \, dx &= \int \sin^2 x \cos^2 x \cos x \, dx \\
&= \int \sin^2 x \, (1 - \sin^2 x) \cos x \, dx \\
&= \int (\sin^2 x - \sin^4 x) \cos x \, dx
\end{aligned}$$

Now let $u = \sin x$. Then $du = \cos x \, dx$ and we get:

$$\begin{aligned}
\int \sin^2 x \cos^3 x \, dx &= \int (\sin^2 x - \sin^4 x) \cos x \, dx \\
&= \int (u^2 - u^4) \, du \\
&= \frac{1}{3}u^3 - \frac{1}{5}u^5 + C \\
&= \boxed{\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C}
\end{aligned}$$

The second integral is computed using the u-substitution method. Let $u = \frac{1}{2}x$. Then $du = \frac{1}{2}\,dx \quad \Rightarrow \quad 2\,du = dx$ and $x = 2u$. Substituting these into the integral and evaluating we get:

$$\begin{aligned}
\int \frac{1}{\sqrt{4-x^2}} \, dx &= \int \frac{1}{\sqrt{4-(2u)^2}} \, (2\,du) \\
&= 2\int \frac{1}{\sqrt{4-4u^2}} \, du \\
&= 2\int \frac{1}{\sqrt{4}\sqrt{1-u^2}} \, du \\
&= \int \frac{1}{\sqrt{1-u^2}} \, du \\
&= \arcsin u + C \\
&= \boxed{\arcsin\left(\frac{1}{2}x\right) + C}
\end{aligned}$$

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Problem 5 Solution

5. Compute the following integrals:

$$\int \frac{x}{\sqrt{x-2}}\,dx \qquad \int \arctan x\,dx$$

Solution: The first integral is computed using the u-substitution method. Let $u = x - 2$. Then $du = dx$ and $x = u + 2$. Substituting these into the integral and evaluating we get:

$$\begin{aligned}
\int \frac{x}{\sqrt{x-2}}\,dx &= \int \frac{u+2}{\sqrt{u}}\,du \\
&= \int \left(u^{1/2} + 2u^{-1/2}\right)\,du \\
&= \frac{2}{3}u^{3/2} + 4u^{1/2} + C \\
&= \boxed{\frac{2}{3}(x-2)^{3/2} + 4(x-2)^{1/2} + C}
\end{aligned}$$

The second integral is computed using Integration by Parts. Let $u = \arctan x$ and $v' = 1$. Then $u' = \dfrac{1}{x^2+1}$ and $v = x$. Using the Integration by Parts formula:

$$\int uv'\,dx = uv - \int u'v\,dx$$

we get:

$$\int \arctan x\,dx = x\arctan x - \int \frac{x}{x^2+1}\,dx$$

The integral on the right hand side is computed using the u-substitution $u = x^2 + 1$. Then $du = 2x\,dx \quad \Rightarrow \quad \dfrac{1}{2}\,du = x\,dx$ and we get:

$$\begin{aligned}
\int \arctan x\,dx &= x\arctan x - \int \frac{x}{x^2+1}\,dx \\
&= x\arctan x - \int \frac{1}{x^2+1}\cdot x\,dx \\
&= x\arctan x - \int \frac{1}{u}\cdot\frac{1}{2}\,du \\
&= x\arctan x - \frac{1}{2}\int \frac{1}{u}\,du \\
&= x\arctan x - \frac{1}{2}\ln|u| + C \\
&= \boxed{x\arctan x - \frac{1}{2}\ln(x^2+1) + C}
\end{aligned}$$

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Problem 6 Solution

6. Compute the following integrals:

$$\int x^3 \sin\left(x^2\right)\, dx, \qquad \int \frac{1}{x^2+x-6}\, dx$$

Solution: The first integral is computed using the u-substitution method. Let $u = x^2$. Then $du = 2x\,dx \quad \Rightarrow \quad \frac{1}{2}\,du = x\,dx$ and we get:

$$\begin{aligned}\int x^3 \sin\left(x^2\right)\, dx &= \int x^2 \sin\left(x^2\right)\, x\,dx \\ &= \int u \sin u \left(\frac{1}{2}\,du\right) \\ &= \frac{1}{2}\int u \sin u\, du\end{aligned}$$

We now use Integration by Parts to evaluate the above integral. Let $w = u$ and $v' = \sin u$. Then $w' = 1$ and $v = -\cos u$. Using the Integration by Parts formula:

$$\int wv'\,du = wv - \int w'v\,du$$

we get:

$$\begin{aligned}\int u \sin u\, du &= u(-\cos u) - \int 1 \cdot (-\cos u)\, du \\ &= -u\cos u + \int \cos u\, du \\ &= -u\cos u + \sin u + C\end{aligned}$$

Therefore,

$$\begin{aligned}\int x^3 \sin\left(x^2\right)\, dx &= \frac{1}{2}\int u \sin u\, du \\ &= \frac{1}{2}\left(-u\cos u + \sin u\right) + C \\ &= -\frac{1}{2}u\cos u + \frac{1}{2}\sin u + C \\ &= \boxed{-\frac{1}{2}x^2\cos\left(x^2\right) + \frac{1}{2}\sin\left(x^2\right) + C}\end{aligned}$$

The second integral is computed using Partial Fraction Decomposition. Factoring the denominator and decomposing we get:

$$\frac{1}{x^2 + x - 6} = \frac{1}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$$

Multiplying the equation by $(x+3)(x-2)$ we get:

$$1 = A(x-2) + B(x+3)$$

Next we plug in two different values of x to get a system of two equations in two unknowns (A, B). Letting $x = -3$ and $x = 2$ we get:

$$\begin{aligned} x = -3: \quad 1 &= A(-3-2) + B(-3+3) \quad \Rightarrow \quad A = -\frac{1}{5} \\ x = 2: \quad 1 &= A(2-2) + B(2+3) \quad \Rightarrow \quad B = \frac{1}{5} \end{aligned}$$

Plugging these values of A and B back into the decomposed equation and integrating we get:

$$\begin{aligned} \int \frac{1}{x^2 + x - 6}\, dx &= \int \left(\frac{-\frac{1}{5}}{x+3} + \frac{\frac{1}{5}}{x-2} \right) dx \\ &= \boxed{-\frac{1}{5} \ln|x+3| + \frac{1}{5} \ln|x-2| + C} \end{aligned}$$

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Problem 7 Solution

7. Compute the following integrals:

$$\int \frac{1}{x^2+2x+3}\,dx, \qquad \int x^6 \ln x\,dx$$

Solution: To compute the first integral, we will first complete the square in the denominator.

$$\int \frac{1}{x^2+2x+3}\,dx = \int \frac{1}{(x+1)^2+2}\,dx$$

Now let $u = x+1$. Then $du = dx$ and we get:

$$\int \frac{1}{(x+1)^2+2}\,dx = \int \frac{1}{u^2+2}\,du$$

Now let $u = \sqrt{2}v$. Then $du = \sqrt{2}\,dv$ and we get:

$$\begin{aligned}
\int \frac{1}{u^2+2}\,du &= \int \frac{1}{(\sqrt{2}v)^2+2}\left(\sqrt{2}\,dv\right) \\
&= \sqrt{2}\int \frac{1}{2v^2+2}\,dv \\
&= \frac{\sqrt{2}}{2}\int \frac{1}{v^2+1}\,dv \\
&= \frac{\sqrt{2}}{2}\arctan v + C \\
&= \frac{\sqrt{2}}{2}\arctan\left(\frac{u}{\sqrt{2}}\right) + C
\end{aligned}$$

Therefore, the original integral is:

$$\int \frac{1}{x^2+2x+3}\,dx = \boxed{\frac{\sqrt{2}}{2}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + C}$$

The second integral is computed using Integration by Parts. Let $u = \ln x$ and $v' = x^6$. Then $u' = \frac{1}{x}$ and $v = \frac{1}{7}x^7$. Using the Integration by Parts formula:

$$\int uv'\,dx = uv - \int u'v\,dx$$

we get:

$$\begin{aligned}\int x^6 \ln x \, dx &= (\ln x)\left(\frac{1}{7}x^7\right) - \int \frac{1}{x} \cdot \frac{1}{7}x^7) \, dx \\ &= \frac{1}{7}x^7 \ln x - \frac{1}{7}\int x^6 \, dx \\ &= \boxed{\frac{1}{7}x^7 \ln x - \frac{1}{49}x^7 + C}\end{aligned}$$

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Problem 8 Solution

8. Compute the following integrals:

$$\int \cos\left(\sqrt{x}\right)\, dx, \qquad \int x^2 e^{2x}\, dx$$

Solution: To begin the solution of the first integral, we first use the u-substitution method. Let $u = \sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}}\, dx \quad \Rightarrow \quad 2u\, du = dx$ and we get:

$$\begin{aligned}\int \cos\left(\sqrt{x}\right)\, dx &= \int \cos u\,(2u\, du)\\ &= 2\int u\cos u\, du\end{aligned}$$

We now use Integration by Parts to evaluate the above integral. Let $w = u$ and $v' = \cos u$. Then $w' = 1$ and $v = \sin u$. Using the Integration by Parts formula:

$$\int wv'\, du = wv - \int w'v\, du$$

we get:

$$\begin{aligned}\int u\cos u\, du &= u\sin u - \int \sin u\, du\\ \int u\cos u\, du &= u\sin u + \cos u + C\end{aligned}$$

Therefore,

$$\begin{aligned}\int \cos\left(\sqrt{x}\right)\, dx &= 2\int u\cos u\, du\\ &= 2\left(u\sin u + \cos u\right) + C\\ &= 2u\sin u + \frac{1}{2}\cos u + C\\ &= \boxed{2\sqrt{x}\sin\left(\sqrt{x}\right) + 2\cos\left(\sqrt{x}\right) + C}\end{aligned}$$

The second integral is computed using Integration by Parts. Let $u = x^2$ and $v' = e^{2x}$. Then $u' = 2x$ and $v = \frac{1}{2}e^{2x}$. Using the Integration by Parts formula:

$$\int uv'\, dx = uv - \int u'v\, dx$$

we get:

$$\int x^2 e^{2x}\,dx = \frac{1}{2}x^2 e^{2x} - \int 2x\left(\frac{1}{2}e^{2x}\right)\,dx$$
$$= \frac{1}{2}x^2 e^{2x} - \int x e^{2x}\,dx$$

A second Integration by Parts must be performed. Let $u = x$ and $v' = e^{2x}$. Then $u' = 1$ and $v = \frac{1}{2}e^{2x}$. Using the Integration by Parts formula again we get:

$$\int x^2 e^{2x}\,dx = \frac{1}{2}x^2 e^{2x} - \left[\frac{1}{2}x e^{2x} - \frac{1}{2}\int e^{2x}\,dx\right]$$
$$= \boxed{\frac{1}{2}x^2 e^{2x} - \frac{1}{2}x e^{2x} + \frac{1}{4}e^{2x} + C}$$

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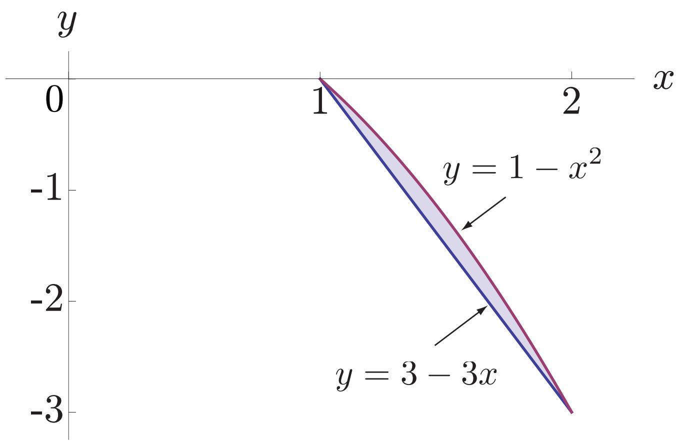
Problem 9 Solution

9. Compute the area enclosed between the graphs $y = 1 - x^2$ and $y = 3 - 3x$.

Solution:



The formula we will use to compute the area of the region is:

$$\text{Area} = \int_a^b (\text{top} - \text{bottom})\, dx$$

where the limits of integration are the x-coordinates of the points of intersection of the two curves. These are found by setting the y's equal to each other and solving for x.

$$\begin{aligned} y &= y \\ 3 - 3x &= 1 - x^2 \\ x^2 - 3x + 2 &= 0 \\ (x-1)(x-2) &= 0 \\ x = 1,\ x &= 2 \end{aligned}$$

From the graph we see that the top curve is $y = 1 - x^2$ and the bottom curve is $y = 3 - 3x$.

Therefore, the area is:

$$
\begin{aligned}
\text{Area} &= \int_1^2 \left[\left(1 - x^2\right) - (3 - 3x)\right]\, dx \\
&= \int_1^2 \left(-2 + 3x - x^2\right)\, dx \\
&= \left[-2x + \frac{3}{2}x^2 - \frac{1}{3}x^3\right]_1^2 \\
&= \left[-2(2) + \frac{3}{2}(2)^2 - \frac{1}{3}(2)^3\right] - \left[-2(1) + \frac{3}{2}(1)^2 - \frac{1}{3}(1)^3\right] \\
&= \left[-4 + 6 - \frac{8}{3}\right] - \left[-2 + \frac{3}{2} - \frac{1}{3}\right] \\
&= \boxed{\frac{1}{6}}
\end{aligned}
$$

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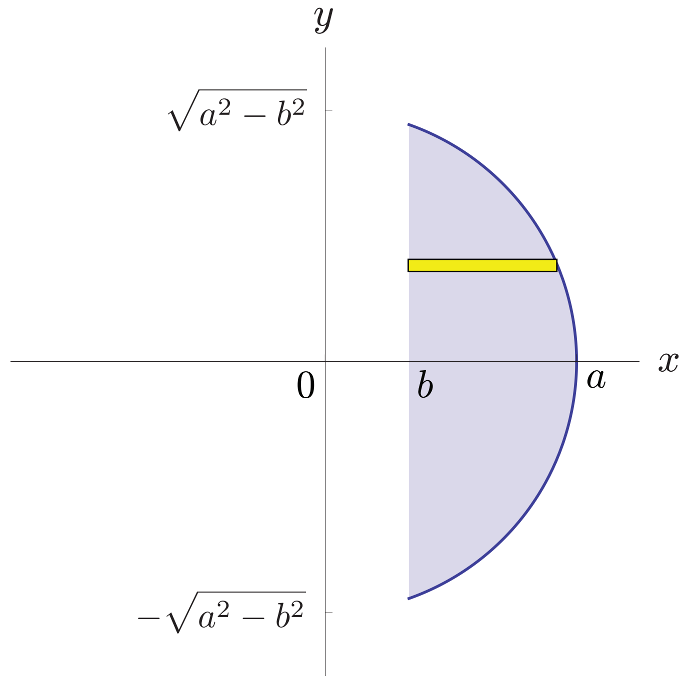
Problem 10 Solution

10. A round hole of radius b is drilled through the center of a hemisphere of radius a ($a > b$). Find the volume of the portion of the sphere that remains.

Solution:



We use the Washer Method to compute the volume. The formula is:

$$V = \pi \int_c^d \left[(\text{right})^2 - (\text{left})^2\right] \, dy$$

where the right curve is $x = \sqrt{a^2 - y^2}$ and the left curve is $x = b$. The limits of integration are the y-coordinates of the points of intersection of the right and left curves. We find these by setting the x's equal to each other and solving for y.

$$\begin{aligned}
x &= x \\
b &= \sqrt{a^2 - y^2} \\
b^2 &= a^2 - y^2 \\
y^2 &= a^2 - b^2 \\
y &= \pm\sqrt{a^2 - b^2}
\end{aligned}$$

The volume is then:

$$
\begin{aligned}
V &= \pi \int_{-\sqrt{a^2-b^2}}^{\sqrt{a^2-b^2}} \left[\left(\sqrt{a^2 - y^2} \right)^2 - b^2 \right] \, dy \\
&= 2\pi \int_0^{\sqrt{a^2-b^2}} \left(a^2 - b^2 - y^2 \right) \, dy \\
&= 2\pi \left[(a^2 - b^2)y - \frac{1}{3}y^3 \right]_0^{\sqrt{a^2-b^2}} \\
&= 2\pi \left[(a^2 - b^2)\sqrt{a^2 - b^2} - \frac{1}{3} \left(\sqrt{a^2 - b^2} \right)^3 \right] \\
&= \boxed{\frac{4\pi}{3}(a^2 - b^2)^{3/2}}
\end{aligned}
$$