Math 181, Exam 2, Fall 2008
Problem 1 Solution

1. Evaluate the integral:

$$\int \frac{dx}{x(x-1)}$$

Solution: We will evaluate the integral using Partial Fraction Decomposition. First, we decompose the rational function into a sum of simpler rational functions.

$$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

Next, we multiply the above equation by $x(x-1)$ to get:

$$1 = A(x-1) + Bx$$

Then we plug in two different values for x to create a system of two equations in two unknowns (A, B). We select $x = 0$ and $x = 1$ for simplicity.

$$\begin{aligned} x = 0: \;\; & A(0-1) + B(0) = 1 \quad \Rightarrow \quad A = -1 \\ x = 1: \;\; & A(1-1) + B(1) = 1 \quad \Rightarrow \quad B = 1 \end{aligned}$$

Finally, we plug these values for A and B back into the decomposition and integrate.

$$\begin{aligned} \int \frac{1}{x(x-1)}\,dx &= \int \left(\frac{A}{x} + \frac{B}{x-1}\right) dx \\ &= \int \left(\frac{-1}{x} + \frac{1}{x-1}\right) dx \\ &= \boxed{-\ln|x| + \ln|x-1| + C} \end{aligned}$$

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Problem 2 Solution

2. Evaluate the integral:

$$\int \sin^2\theta \cos^3\theta \, d\theta$$

Solution: The integral is computed by rewriting the integral using the Pythagorean Identity $\cos^2\theta + \sin^2\theta = 1$.

$$\begin{aligned}\int \sin^2\theta \cos^3\theta \, d\theta &= \int \sin^2\theta \cos^2\theta \cos\theta \, d\theta \\ &= \int \sin^2\theta \, (1 - \sin^2\theta) \cos\theta \, d\theta \\ &= \int (\sin^2\theta - \sin^4\theta) \cos\theta \, d\theta\end{aligned}$$

Now let $u = \sin\theta$. Then $du = \cos\theta \, d\theta$ and we get:

$$\begin{aligned}\int \sin^2\theta \cos^3\theta \, d\theta &= \int (\sin^2\theta - \sin^4\theta) \cos\theta \, d\theta \\ &= \int (u^2 - u^4) \, du \\ &= \frac{1}{3}u^3 - \frac{1}{5}u^5 + C \\ &= \boxed{\frac{1}{3}\sin^3\theta - \frac{1}{5}\sin^5\theta + C}\end{aligned}$$

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Problem 3 Solution

3. Evaluate the integral:

$$\int x^2 \ln x \, dx$$

Solution: We evaluate the integral using Integration by Parts. Let $u = \ln x$ and $v' = x^2$. Then $u' = \dfrac{1}{x}$ and $v = \dfrac{1}{3}x^3$. Using the Integration by Parts formula:

$$\int uv' \, dx = uv - \int u'v \, dx$$

we get:

$$\begin{aligned}
\int x^2 \ln x \, dx &= (\ln x)\left(\frac{1}{3}x^3\right) - \int \frac{1}{x} \cdot \frac{1}{3}x^3 \, dx \\
&= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 \, dx \\
&= \boxed{\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C}
\end{aligned}$$

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Problem 4 Solution

4. Evaluate the integral:

$$\int \frac{\sqrt{1-x^2}}{x^2}\,dx$$

Solution: To evaluate the integral we use the trigonometric substitution $x = \sin\theta$. Then $dx = \cos\theta\,d\theta$ and we get:

$$\begin{aligned}
\int \frac{\sqrt{1-x^2}}{x^2}\,dx &= \int \frac{\sqrt{1-\sin^2\theta}}{\sin^2\theta}\,(\cos\theta\,d\theta) \\
&= \int \frac{\cos\theta}{\sin^2\theta}\,\cos\theta\,d\theta \\
&= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta \\
&= \int \cot^2\theta\,d\theta \\
&= \int \left(\csc^2\theta - 1\right)\,d\theta \\
&= \int \csc^2\theta\,d\theta - \int 1\,d\theta \\
&= -\cot\theta - \theta + C
\end{aligned}$$

We used the identity $1 + \cot^2\theta = \csc^2\theta$ and the fact that $(\cot\theta)' = -\csc^2\theta$ to get to the answer. Now use the fact that $\sin\theta = x$ and $\cos\theta = \sqrt{1-x^2}$ to write the answer in terms of x.

$$\begin{aligned}
\int \frac{\sqrt{1-x^2}}{x^2}\,dx &= -\cot\theta - \theta + C \\
&= -\frac{\cos\theta}{\sin\theta} - \theta + C \\
&= \boxed{-\frac{\sqrt{1-x^2}}{x} - \arcsin x + C}
\end{aligned}$$

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Problem 5 Solution

5. Calculate the arc length of $y = (x-1)^{3/2} + 2$ over the interval $[1, 6]$.

Solution: The arclength is:

$$\begin{aligned} L &= \int_a^b \sqrt{1 + f'(x)^2}\, dx \\ &= \int_1^6 \sqrt{1 + \left(\frac{3}{2}(x-1)^{1/2}\right)^2}\, dx \\ &= \int_1^6 \sqrt{1 + \frac{9}{4}(x-1)}\, dx \\ &= \int_1^6 \sqrt{\frac{9}{4}x - \frac{5}{4}}\, dx \\ &= \frac{1}{2}\int_1^6 \sqrt{9x-5}\, dx \end{aligned}$$

We now use the u-substitution $u = 9x - 5$. Then $\frac{1}{9}\,du = dx$, the lower limit of integration changes from 1 to 4, and the upper limit of integration changes from 6 to 49.

$$\begin{aligned} L &= \frac{1}{2}\int_1^6 \sqrt{9x-5}\, dx \\ &= \frac{1}{18}\int_4^{49} \sqrt{u}\, du \\ &= \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_4^{49} \\ &= \frac{1}{18}\left[\frac{2}{3}(49)^{3/2} - \frac{2}{3}(4)^{3/2}\right] \\ &= \frac{1}{27}\Big[343 - 8\Big] \\ &= \boxed{\frac{335}{27}} \end{aligned}$$

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Problem 6 Solution

6. Either compute the value of the following integral or show that it diverges.

$$\int_0^\infty xe^{-x^2}\,dx$$

Solution: We evaluate the integral by turning it into a limit calculation.

$$\int_0^\infty xe^{-x^2}\,dx = \lim_{R\to\infty}\int_0^R xe^{-x^2}\,dx$$

We use the u-substitution to compute the integral. Let $u = -x^2$ and $-\frac{1}{2}du = x\,dx$. The indefinite integral is then:

$$\int xe^{-x^2}\,dx = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u = -\frac{1}{2}e^{-x^2}$$

The definite integral from 0 to R is:

$$\begin{aligned}\int_0^R xe^{-x^2}\,dx &= \left[-\frac{1}{2}e^{-x^2}\right]_0^R \\ &= -\frac{1}{2}e^{-R^2} + \frac{1}{2}e^{-0^2} \\ &= -\frac{1}{2e^{R^2}} + \frac{1}{2}\end{aligned}$$

Taking the limit as $R \to \infty$ we get:

$$\begin{aligned}\int_0^\infty xe^{-x^2}\,dx &= \lim_{R\to\infty}\int_0^R xe^{-x^2}\,dx \\ &= \lim_{R\to\infty}\left(-\frac{1}{2e^{R^2}} + \frac{1}{2}\right) \\ &= -0 + \frac{1}{2} \\ &= \boxed{\frac{1}{2}}\end{aligned}$$

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Problem 7 Solution

7. Compute the 3rd degree Taylor polynomial for the function $f(x) = \sin(2x)$ centered at $x = \frac{\pi}{2}$.

Solution: The 3rd degree Taylor polynomial $T_3(x)$ of $f(x)$ centered at $x = \frac{\pi}{2}$ has the formula:

$$T_3(x) = f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right) + \frac{f''(\frac{\pi}{2})}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{f'''(\frac{\pi}{2})}{3!}\left(x - \frac{\pi}{2}\right)^3$$

The derivatives of $f(x)$ and their values at $x = \frac{\pi}{2}$ are:

k	$f^{(k)}(x)$	$f^{(k)}(\frac{\pi}{2})$
0	$\sin(2x)$	$\sin(2 \cdot \frac{\pi}{2}) = 0$
1	$2\cos(2x)$	$2\cos(2 \cdot \frac{\pi}{2}) = -2$
2	$-4\sin(2x)$	$-4\sin(2 \cdot \frac{\pi}{2}) = 0$
3	$-8\cos(2x)$	$-8\cos(2 \cdot \frac{\pi}{2}) = 8$

The function $T_3(x)$ is then:

$$T_3(x) = f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right) + \frac{f''(\frac{\pi}{2})}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{f'''(\frac{\pi}{2})}{3!}\left(x - \frac{\pi}{2}\right)^3$$

$$T_3(x) = 0 - 2\left(x - \frac{\pi}{2}\right) + \frac{0}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{8}{3!}\left(x - \frac{\pi}{2}\right)^3$$

$$\boxed{T_3(x) = -2\left(x - \frac{\pi}{2}\right) + \frac{4}{3}\left(x - \frac{\pi}{2}\right)^3}$$