Math 181, Exam 2, Fall 2009
Problem 1 Solution

1. Compute the integral: $\displaystyle\int_0^{+\infty} xe^{-x^2}\,dx$.

Solution: We evaluate the integral by turning it into a limit calculation.

$$\int_0^{+\infty} xe^{-x^2}\,dx = \lim_{R\to+\infty}\int_0^R xe^{-x^2}\,dx$$

We use the u-substitution to compute the integral. Let $u = -x^2$ and $-\frac{1}{2}du = x\,dx$. The indefinite integral is then:

$$\int xe^{-x^2}\,dx = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u = -\frac{1}{2}e^{-x^2}$$

The definite integral from 0 to R is:

$$\begin{aligned}\int_0^R xe^{-x^2}\,dx &= \left[-\frac{1}{2}e^{-x^2}\right]_0^R \\ &= -\frac{1}{2}e^{-R^2} + \frac{1}{2}e^{-0^2} \\ &= -\frac{1}{2e^{R^2}} + \frac{1}{2}\end{aligned}$$

Taking the limit as $R \to +\infty$ we get:

$$\begin{aligned}\int_0^{+\infty} xe^{-x^2}\,dx &= \lim_{R\to+\infty}\int_0^R xe^{-x^2}\,dx \\ &= \lim_{R\to+\infty}\left(-\frac{1}{2e^{R^2}} + \frac{1}{2}\right) \\ &= -0 + \frac{1}{2} \\ &= \boxed{\frac{1}{2}}\end{aligned}$$

Math 181, Exam 2, Fall 2009
Problem 2 Solution

2. Determine whether the integral $\displaystyle\int_1^{+\infty} \frac{x\,dx}{\sqrt{x^5+x+1}}\,dx$ converges or not.

Solution: We use the Comparison Test to show that the integral converges. Let $g(x) = \frac{x}{\sqrt{x^5+x+1}}$. We must find a function $f(x)$ such that:

$$(1)\ \int_1^{+\infty} f(x)\,dx \text{ converges} \quad \text{and} \quad (2)\ \ 0 \le g(x) \le f(x) \text{ for } x \ge 1.$$

We choose $f(x) = \frac{x}{\sqrt{x^5}} = \frac{x}{x^{5/2}} = \frac{1}{x^{3/2}}$. This choice of $f(x)$ satisfies the inequality

$$0 \le g(x) \le f(x)$$
$$0 \le \frac{x}{\sqrt{x^5+x+1}} \le \frac{x}{\sqrt{x^5}} = \frac{1}{x^{3/2}}$$

for $x \ge 1$ using the argument that the denominator of $g(x)$ is larger than the denominator of $f(x)$ for these values of x. Furthermore, the integral $\int_1^{+\infty} f(x)\,dx = \int_1^{+\infty} \frac{1}{x^{3/2}}\,dx$ converges because it is a p-integral with $p = \frac{3}{2} > 1$. Therefore, the integral $\int_1^{+\infty} g(x)\,dx = \int_1^{+\infty} \frac{x\,dx}{\sqrt{x^5+x+1}}$ **converges** by the Comparison Test.

Math 181, Exam 2, Fall 2009
Problem 3a Solution

3a. Find the length of the graph of the function $f(x) = e^{x/2} + e^{-x/2} + 2$ from $x = 0$ to $x = \ln 2$.

Solution: The arclength is:

$$
\begin{aligned}
L &= \int_a^b \sqrt{1 + f'(x)^2}\,dx \\
&= \int_0^{\ln 2} \sqrt{1 + \left(\frac{1}{2}e^{x/2} - \frac{1}{2}e^{-x/2}\right)^2}\,dx \\
&= \int_0^{\ln 2} \sqrt{1 + \frac{1}{4}e^{x} - \frac{1}{2} + \frac{1}{4}e^{-x}}\,dx \\
&= \int_0^{\ln 2} \sqrt{\frac{1}{4}e^{x} + \frac{1}{2} + \frac{1}{4}e^{-x}}\,dx \\
&= \int_0^{\ln 2} \sqrt{\left(\frac{1}{2}e^{x/2} + \frac{1}{2}e^{-x/2}\right)^2}\,dx \\
&= \int_0^{\ln 2} \left(\frac{1}{2}e^{x/2} + \frac{1}{2}e^{-x/2}\right)\,dx \\
&= \frac{1}{2}\int_0^{\ln 2} \left(e^{x/2} + e^{-x/2}\right)\,dx \\
&= \frac{1}{2}\left[2e^{x/2} - 2e^{-x/2}\right]_0^{\ln 2} \\
&= \left[e^{x/2} - e^{-x/2}\right]_0^{\ln 2} \\
&= \left[e^{\ln 2/2} - e^{-\ln 2/2}\right] - \left[e^{0/2} - e^{-0/2}\right] \\
&= \left[\sqrt{2} - \frac{1}{\sqrt{2}}\right] - \left[1 - 1\right] \\
&= \boxed{\sqrt{2} - \frac{1}{\sqrt{2}}}
\end{aligned}
$$

Math 181, Exam 2, Fall 2009
Problem 3b Solution

3b. Find the centroid, $(\bar{x}, \bar{y})$, of the region above the x-axis ($y \geq 0$), below the graph of $y = 4 - x^2$, and to the right of the y-axis ($x \geq 0$).

Solution: The coordinates of the centroid are given by the formulas:

$$\bar{x} = \frac{M_y}{M}, \qquad \bar{y} = \frac{M_x}{M}$$

where

$$M_x = \frac{1}{2}\int_a^b f(x)^2\,dx \qquad M_y = \int_a^b x f(x)\,dx \qquad M = \int_a^b f(x)\,dx$$

In the formulas, we use $a = 0$, $b = 2$, and $f(x) = 4 - x^2$ to get:

$$\begin{aligned} M_x &= \frac{1}{2}\int_0^2 \left(4 - x^2\right)^2\,dx \\ &= \frac{1}{2}\int_0^2 \left(16 - 8x^2 + x^4\right)\,dx \\ &= \frac{1}{2}\left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_0^2 \\ &= \frac{128}{15} \end{aligned}$$

$$\begin{aligned} M_y &= \int_0^2 x\left(4 - x^2\right)\,dx \\ &= \int_0^2 \left(4x - x^3\right)\,dx \\ &= \left[2x^2 - \frac{1}{4}x^4\right]_0^2 \\ &= 4 \end{aligned}$$

$$\begin{aligned} M &= \int_0^2 \left(4 - x^2\right)\,dx \\ &= \left[4x - \frac{1}{3}x^3\right]_0^2 \\ &= \frac{16}{3} \end{aligned}$$

Therefore, the centroid has coordinates:

$$\begin{aligned} \bar{x} &= \frac{M_y}{M} = \frac{4}{\frac{16}{3}} = \boxed{\frac{3}{4}} \\ \bar{y} &= \frac{M_x}{M} = \frac{\frac{128}{15}}{\frac{16}{3}} = \boxed{\frac{8}{5}} \end{aligned}$$

Math 181, Exam 2, Fall 2009
Problem 4 Solution

4. Find the 3rd Taylor polynomial of the function $f(x) = x^3 + 2x^2 + x - 2$ centered at $a = 1$.

Solution: The 3rd degree Taylor polynomial $T_3(x)$ of $f(x)$ centered at $a = 1$ has the formula:

$$T_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

The derivatives of $f(x)$ and their values at $x = 1$ are:

k	$f^{(k)}(x)$	$f^{(k)}(1)$
0	$x^3 + 2x^2 + x - 2$	$1^3 + 2(1)^2 + 1 - 2 = 2$
1	$3x^2 + 4x + 1$	$3(1)^2 + 4(1) + 1 = 8$
2	$6x + 4$	$6(1) + 4 = 10$
3	6	6

The function $T_3(x)$ is then:

$$T_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$
$$T_3(x) = 2 + 8(x-1) + \frac{10}{2!}(x-1)^2 + \frac{6}{3!}(x-1)^3$$
$$\boxed{T_3(x) = 2 + 8(x-1) + 5(x-1)^2 + (x-1)^3}$$

Math 181, Exam 2, Fall 2009
Problem 5 Solution

5. Compute the sum of the series: $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^n 3^{n-1}}{4^{n+3}}$.

Solution: We recognize the given series as a geometric series. In order to find its sum we must first rewrite the series.

$$\sum_{n=1}^{+\infty} \frac{(-1)^n 3^{n-1}}{4^{n+3}} = \sum_{n=1}^{+\infty} \frac{(-1)^n 3^n 3^{-1}}{4^n 4^3} = \sum_{n=1}^{+\infty} \frac{3^{-1}}{4^3} \cdot \frac{(-1)^n 3^n}{4^n} = \sum_{n=1}^{+\infty} \frac{1}{192} \left(-\frac{3}{4}\right)^n$$

This is a convergent geometric series because $|r| = |-\frac{3}{4}| < 1$. We can now use the formula:

$$\sum_{n=M}^{+\infty} cr^n = r^M \cdot \frac{c}{1-r}$$

where $M = 1$, $c = \frac{1}{192}$, and $r = -\frac{3}{4}$. The sum of the series is then:

$$\sum_{n=1}^{+\infty} \frac{1}{192} \left(-\frac{3}{4}\right)^n = \left(-\frac{3}{4}\right)^1 \cdot \frac{\frac{1}{192}}{1 - (-\frac{3}{4})} = \boxed{-\frac{1}{448}}$$

Math 181, Exam 2, Fall 2009
Problem 6 Solution

6. Determine whether the following series converge or not:

$$\sum_{n=1}^{+\infty} \frac{(-1)^n}{\sqrt{n^2+1}}, \qquad \sum_{n=2}^{+\infty} \frac{1}{n(\ln n)^2}, \qquad \sum_{n=1}^{+\infty} n3^{-n}$$

Solution: The first series is alternating so we use the Leibniz Test determine whether or not it converges. Let $a_n = f(n) = \frac{1}{\sqrt{n^2+1}}$. The function $f(n)$ is decreasing for $n \geq 1$ and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} = 0$$

Therefore, the series $\sum_{n=1}^{+\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$ **converges** by the Leibniz Test.

We use the Integral Test to determine whether or not the second series converges. Let $f(x) = \frac{1}{x(\ln x)^2}$. The function $f(x)$ is decreasing for $x \geq 2$. We must now determine whether or not the following integral converges:

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{R\to\infty} \int_2^R \frac{1}{x(\ln x)^2}\,dx$$

Let $u = \ln x$. Then $du = \frac{1}{x}\,dx$ and we get:

$$\begin{aligned}
\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx &= \lim_{R\to\infty} \int_2^R \frac{1}{x(\ln x)^2}\,dx \\
&= \lim_{R\to\infty} \int_{\ln 2}^{\ln R} \frac{1}{u^2}\,du \\
&= \lim_{R\to\infty} \left[-\frac{1}{u}\right]_{\ln 2}^{\ln R} \\
&= \lim_{R\to\infty} \left(\frac{1}{\ln 2} - \frac{1}{\ln R}\right) \\
&= \frac{1}{\ln 2}
\end{aligned}$$

Since the integral converges, the series $\sum_{n=2}^{+\infty} \frac{1}{n(\ln n)^2}$ **converges** by the Integral Test.

The third series can be rewritten as:

$$\sum_{n=1}^{+\infty} n3^{-n} = \sum_{n=1}^{+\infty} \frac{n}{3^n}$$

We use the Ratio Test to determine whether or not this series converges.

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\
&= \lim_{n\to\infty} \frac{n+1}{3^{n+1}} \cdot \frac{3^n}{n} \\
&= \lim_{n\to\infty} \frac{1}{3}\left(\frac{n+1}{n}\right) \\
&= \lim_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right) \\
&= \frac{1}{3}
\end{aligned}$$

Since $\rho = \frac{1}{3} < 1$, the series $\sum_{n=1}^{+\infty} n3^{-n}$ **converges** by the Ratio Test.