Math 181, Exam 2, Spring 2009
Problem 1 Solution

1. Compute the integrals:

$$\int \arctan x \, dx \qquad \int \frac{dx}{x^3 + x}$$

Solution: We will evaluate the first integral using Integration by Parts. Let $u = \arctan x$ and $v' = 1$. Then $u' = \dfrac{1}{x^2+1}$ and $v = x$. Using the Integration by Parts formula:

$$\int uv' \, dx = uv - \int u'v \, dx$$

we get:

$$\int \arctan x \, dx = x \arctan x - \int \frac{1}{x^2+1} \, x \, dx.$$

Use the substitution $w = x^2 + 1$ to evaluate the integral on the right hand side. Then $dw = 2x \, dx \quad \Rightarrow \quad \frac{1}{2} dw = x \, dx$ and we get:

$$\begin{aligned} \int \arctan x \, dx &= x \arctan x - \frac{1}{2} \int \frac{1}{w} \, dw \\ &= x \arctan x - \frac{1}{2} \ln |w| + C \\ &= \boxed{x \arctan x - \frac{1}{2} \ln(x^2+1) + C} \end{aligned}$$

Note that the absolute value signs aren't needed because $x^2 + 1 > 0$ for all x.

We will evaluate the second integral using Partial Fraction Decomposition. First, we factor the denominator and then decompose the rational function into a sum of simpler rational functions.

$$\frac{1}{x^3+x} = \frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Next, we multiply the above equation by $x(x^2+1)$ to get:

$$1 = A(x^2+1) + (Bx+C)x$$

Then we plug in three different values for x to create a system of three equations in three unknowns (A, B, C). We select $x = 0$, $x = 1$, and $x = -1$ for simplicity.

$$\begin{aligned} x = 0: &\quad A(0^2+1) + (B(0)+C)(0) = 1 \quad \Rightarrow \quad A = 1 \\ x = 1: &\quad (1)(1^2+1) + (B(1)+C)(1) = 1 \quad \Rightarrow \quad B + C = -1 \\ x = -1: &\quad (1)((-1)^2+1) + (B(-1)+C)(-1) = 1 \quad \Rightarrow \quad B - C = -1 \end{aligned}$$

The solution to this system is $A = 1$, $B = -1$ and $C = 0$. Finally, we plug these values for A, B, and C back into the decomposition and integrate.

$$\begin{aligned}
\int \frac{dx}{x^3 + x} &= \int \left(\frac{1}{x} + \frac{-x + 0}{x^2 + 1} \right) dx \\
&= \int \frac{1}{x}\, dx - \int \frac{x}{x^2 + 1}\, dx \\
&= \boxed{\ln|x| - \frac{1}{2}\ln(x^2 + 1) + C}
\end{aligned}$$

We solved the integral $\displaystyle\int \frac{x}{x^2 + 1}\, dx$ in the first part of the problem above.

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Problem 2 Solution

2. Compute the integrals:

$$\int \frac{dx}{(1-x^2)^{3/2}} \qquad \int \frac{x\,dx}{(x-1)^3}$$

Solution: We evaluate the first integral using the trigonometric substitution $x = \sin\theta$. Then $dx = \cos\theta\,d\theta$ and we get:

$$\begin{aligned}
\int \frac{dx}{(1-x^2)^{3/2}} &= \int \frac{\cos\theta\,d\theta}{(1-\sin^2\theta)^{3/2}} \\
&= \int \frac{\cos\theta}{(\cos^2\theta)^{3/2}}\,d\theta \\
&= \int \frac{\cos\theta}{\cos^3\theta}\,d\theta \\
&= \int \frac{1}{\cos^2\theta}\,d\theta \\
&= \int \sec^2\theta\,d\theta \\
&= \tan\theta + C
\end{aligned}$$

Since $x = \sin\theta$ we know that $\cos\theta = \sqrt{1-x^2}$ after using the Pythagorean Identity $\sin^2\theta + \cos^2\theta = 1$. We can now rewrite $\tan\theta$ in terms of x.

$$\begin{aligned}
\int \frac{dx}{(1-x^2)^{3/2}} &= \tan\theta + C \\
&= \frac{\sin\theta}{\cos\theta} + C \\
&= \boxed{\frac{x}{\sqrt{1-x^2}} + C}
\end{aligned}$$

To evaluate the second integral we use the u-substitution method. Let $u = x - 1$. Then

$du = dx$ and $x = u + 1$ and we get:

$$\begin{aligned}
\int \frac{x\,dx}{(x-1)^3} &= \int \frac{(u+1)\,du}{u^3} \\
&= \int \left(\frac{u}{u^3} + \frac{1}{u^3} \right) du \\
&= \int \left(u^{-2} + u^{-3} \right) du \\
&= -u^{-1} - \frac{1}{2}u^{-2} + C \\
&= -\frac{1}{u} - \frac{1}{2u^2} + C \\
&= \boxed{-\frac{1}{x-1} - \frac{1}{2(x-1)^2} + C}
\end{aligned}$$

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Problem 3 Solution

3. Compute the improper integral:

$$\int_{\ln 2}^{+\infty} xe^{-x}\,dx$$

Solution: We evaluate the integral by turning it into a limit calculation.

$$\int_{\ln 2}^{+\infty} xe^{-x}\,dx = \lim_{R\to+\infty} \int_{\ln 2}^{R} xe^{-x}\,dx$$

We use Integration by Parts to compute the integral. Let $u = x$ and $v' = e^{-x}$. Then $u' = 1$ and $v = -e^{-x}$. Using the Integration by Parts formula we get:

$$\begin{aligned}
\int_a^b uv'\,dx &= \Big[uv\Big]_a^b - \int_a^b u'v\,dx \\
\int_{\ln 2}^{R} xe^{-x}\,dx &= \Big[-xe^{-x}\Big]_{\ln 2}^{R} - \int_{\ln 2}^{R} \left(-e^{-x}\right)\,dx \\
&= \Big[-xe^{-x}\Big]_{\ln 2}^{R} + \int_{\ln 2}^{R} e^{-x}\,dx \\
&= \Big[-xe^{-x}\Big]_{\ln 2}^{R} + \Big[-e^{-x}\Big]_{\ln 2}^{R} \\
&= \Big[-Re^{-R} + (\ln 2)e^{-\ln 2}\Big] + \Big[-e^{-R} + e^{-\ln 2}\Big] \\
&= -\frac{R}{e^R} + \frac{\ln 2}{e^{\ln 2}} - \frac{1}{e^R} + \frac{1}{e^{\ln 2}} \\
&= -\frac{R}{e^R} + \frac{\ln 2}{2} - \frac{1}{e^R} + \frac{1}{2}
\end{aligned}$$

We now take the limit of the above function as $R \to +\infty$.

$$\begin{aligned}
\int_{\ln 2}^{+\infty} xe^{-x}\,dx &= \lim_{R\to+\infty} \int_{\ln 2}^{R} xe^{-x}\,dx \\
&= \lim_{R\to+\infty} \left(-\frac{R}{e^R} + \frac{\ln 2}{2} - \frac{1}{e^R} + \frac{1}{2}\right) \\
&= -\lim_{R\to+\infty} \frac{R}{e^R} + \frac{\ln 2}{2} - \lim_{R\to+\infty} \frac{1}{e^R} + \frac{1}{2} \\
&= -\lim_{R\to+\infty} \frac{R}{e^R} + \frac{\ln 2}{2} - 0 + \frac{1}{2} \\
&\overset{\text{L'H}}{=} -\lim_{R\to+\infty} \frac{(R)'}{(e^R)'} + \frac{\ln 2}{2} - 0 + \frac{1}{2} \\
&= -\lim_{R\to+\infty} \frac{1}{e^R} + \frac{\ln 2}{2} - 0 + \frac{1}{2} \\
&= -0 + \frac{\ln 2}{2} - 0 + \frac{1}{2} \\
&= \boxed{\frac{\ln 2}{2} + \frac{1}{2}}
\end{aligned}$$

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Problem 4 Solution

4. Compute the arclength of the graph of $y = 2x^{3/2}$ from $x = 0$ to $x = 1$.

Solution: The arclength is:

$$\begin{aligned} L &= \int_a^b \sqrt{1 + f'(x)^2}\, dx \\ &= \int_0^1 \sqrt{1 + \left(3x^{1/2}\right)^2}\, dx \\ &= \int_0^1 \sqrt{1 + 9x}\, dx \end{aligned}$$

We now use the u-substitution $u = 1 + 9x$. Then $\frac{1}{9}\, du = dx$, the lower limit of integration changes from 0 to 1, and the upper limit of integration changes from 1 to 10.

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + 9x}\, dx \\ &= \frac{1}{9} \int_1^{10} \sqrt{u}\, du \\ &= \frac{1}{9} \left[\frac{2}{3} u^{3/2} \right]_1^{10} \\ &= \frac{1}{9} \left[\frac{2}{3}(10)^{3/2} - \frac{2}{3}(1)^{3/2} \right] \\ &= \boxed{\frac{2}{27} \left[10^{3/2} - 1 \right]} \end{aligned}$$

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Problem 5 Solution

5. Find the 3rd Maclaurin polynomial of the function $f(x) = 2\sin(3x)$.

Solution: The 3rd degree Maclaurin polynomial $T_3(x)$ of $f(x)$ has the formula:

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

The derivatives of $f(x)$ and their values at $x = 0$ are:

k	$f^{(k)}(x)$	$f^{(k)}(0)$
0	$2\sin(3x)$	$2\sin(3\cdot 0) = 0$
1	$6\cos(3x)$	$6\cos(3\cdot 0) = 6$
2	$-18\sin(3x)$	$-18\sin(3\cdot 0) = 0$
3	$-54\cos(3x)$	$-54\cos(3\cdot 0) = -54$

The function $T_3(x)$ is then:

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$
$$T_3(x) = 0 + 6x + \frac{0}{2!}x^2 - \frac{54}{3!}x^3$$
$$\boxed{T_3(x) = 6x - 9x^3}$$