Math 181, Final Exam, Fall 2008
Problem 1 Solution

1. Evaluate:

(a) $\int x^2 \cos(x^3)\, dx$ (b) $\int \frac{dx}{x^2 - 5x + 4}$ (c) $\int \frac{dx}{4x^2 + 1}$ (d) $\int \sin^5 x\, dx$

Solution:

(a) We solve the integral using the u-substitution method. Let $u = x^3$ and $\frac{1}{3}\,du = x^2\,dx$. Then we have:

$$\begin{aligned} \int x^2 \cos(x^3)\, dx &= \frac{1}{3}\int \cos u\, du \\ &= \frac{1}{3}\sin u + C \\ &= \boxed{\frac{1}{3}\sin(x^3) + C} \end{aligned}$$

(b) We will evaluate the integral using Partial Fraction Decomposition. First, we factor the denominator and then decompose the rational function into a sum of simpler rational functions.

$$\frac{1}{x^2 - 5x + 4} = \frac{1}{(x-1)(x-4)} = \frac{A}{x-1} + \frac{B}{x-4}$$

Next, we multiply the above equation by $(x-1)(x-4)$ to get:

$$1 = A(x-4) + B(x-1)$$

Then we plug in two different values for x to create a system of two equations in two unknowns (A, B). We select $x = 1$ and $x = 4$ for simplicity.

$$\begin{aligned} x = 1:\ \ A(1-4) + B(1-1) = 1 \quad &\Rightarrow \quad A = -\frac{1}{3} \\ x = 4:\ \ A(4-4) + B(4-1) = 1 \quad &\Rightarrow \quad B = \frac{1}{3} \end{aligned}$$

Finally, we plug these values for A and B back into the decomposition and integrate.

$$\begin{aligned} \int \frac{1}{x^2 - 5x + 4}\, dx &= \int \left(\frac{A}{x-1} + \frac{B}{x-4}\right) dx \\ &= \int \left(\frac{-\frac{1}{3}}{x-1} + \frac{\frac{1}{3}}{x-4}\right) dx \\ &= \boxed{-\frac{1}{3}\ln|x-1| + \frac{1}{3}\ln|x-4| + C} \end{aligned}$$

(c) We solve the integral using the u-substitution method. Let $\frac{1}{2}u = x$ and $\frac{1}{2}\,du = dx$. Then we have:

$$\begin{aligned}\int \frac{dx}{4x^2+1} &= \frac{1}{2}\int \frac{du}{u^2+1} \\ &= \frac{1}{2}\arctan u + C \\ &= \boxed{\frac{1}{2}\arctan(2x) + C}\end{aligned}$$

(d) We first rewrite the integral as follows:

$$\begin{aligned}\int \sin^5 x\,dx &= \int \sin^4 x \sin x\,dx \\ &= \int (\sin^2 x)^2 \sin x\,dx \\ &= \int (1-\cos^2 x)^2 \sin x\,dx\end{aligned}$$

Now let $u = \cos x$. Then $-\,du = \sin x\,dx$ and we have:

$$\begin{aligned}\int \sin^5 x\,dx &= \int (1-\cos^2 x)^2 \sin x\,dx \\ &= -\int (1-u^2)^2\,du \\ &= -\int (1-2u^2+u^4)\,du \\ &= -\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right) + C \\ &= \boxed{-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C}\end{aligned}$$

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Problem 2 Solution

2. Differentiate the function

$$F(x) = \int_2^{x^2} \sin(t^2)\,dt$$

Solution: Using the Fundamental Theorem of Calculus Part II and the Chain Rule, the derivative of $F(x) = \displaystyle\int_a^{u(x)} f(t)\,dt$ is:

$$\begin{aligned} F'(x) &= \frac{d}{dx}\int_a^{u(x)} f(t)\,dt \\ &= f(u(x)) \cdot \frac{d}{dx}u(x) \end{aligned}$$

Applying the formula to the given function $F(x)$ we get:

$$\begin{aligned} F'(x) &= \frac{d}{dx}\int_2^{x^2} \sin(t^2)\,dt \\ &= \sin((x^2)^2) \cdot \frac{d}{dx}(x^2) \\ &= \boxed{\sin(x^4) \cdot (2x)} \end{aligned}$$

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Problem 3 Solution

3. Estimate the value of $\displaystyle\int_0^2 \frac{dx}{x+1}$ using: Estimate the value of the integral using:

(a) the Midpoint method with $N = 2$

(b) the Trapezoidal Rule with $N = 2$

Write your answers as a single, reduced fraction.

Solution:

(a) The length of each subinterval of $[0, 2]$ is:

$$\Delta x = \frac{b-a}{N} = \frac{2-0}{2} = 1$$

The estimate M_2 is:

$$\begin{aligned} M_2 &= \Delta x \left[f\left(\frac{1}{2}\right) + f\left(\frac{3}{2}\right) \right] \\ &= 1 \cdot \left[\frac{1}{\frac{1}{2}+1} + \frac{1}{\frac{3}{2}+1} \right] \\ &= \frac{2}{3} + \frac{2}{5} \\ &= \boxed{\frac{16}{15}} \end{aligned}$$

(b) The length of each subinterval of $[0, 2]$ is $\Delta x = 1$ just as in part (a). The estimate T_2 is:

$$\begin{aligned} T_2 &= \frac{\Delta x}{2} \left[f(0) + 2f(1) + f(2) \right] \\ &= \frac{1}{2} \left[\frac{1}{0+1} + 2 \cdot \frac{1}{1+1} + \frac{1}{2+1} \right] \\ &= \frac{1}{2} \left[1 + 1 + \frac{1}{3} \right] \\ &= \boxed{\frac{7}{6}} \end{aligned}$$

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Problem 4 Solution

4. Evaluate each improper integral:

(a) $\displaystyle\int_0^{+\infty} xe^{-x}\,dx$ (b) $\displaystyle\int_1^2 \frac{dx}{\sqrt{x-1}}$

Solution:

(a) We evaluate the first integral by turning it into a limit calculation.

$$\int_0^{+\infty} xe^{-x}\,dx = \lim_{R\to+\infty}\int_0^R xe^{-x}\,dx$$

We use Integration by Parts to compute the integral. Let $u = x$ and $v' = e^{-x}$. Then $u' = 1$ and $v = -e^{-x}$. Using the Integration by Parts formula we get:

$$\begin{aligned}
\int_a^b uv'\,dx &= \Big[uv\Big]_a^b - \int_a^b u'v\,dx \\
\int_0^R xe^{-x}\,dx &= \Big[-xe^{-x}\Big]_0^R - \int_0^R \left(-e^{-x}\right)\,dx \\
&= \Big[-xe^{-x}\Big]_0^R + \int_0^R e^{-x}\,dx \\
&= \Big[-xe^{-x}\Big]_0^R + \Big[-e^{-x}\Big]_0^R \\
&= \Big[-Re^{-R} + 0e^{-0}\Big] + \Big[-e^{-R} + e^{-0}\Big] \\
&= -\frac{R}{e^R} - \frac{1}{e^R} + 1
\end{aligned}$$

We now take the limit of the above function as $R \to +\infty$.

$$\begin{aligned}
\int_0^{+\infty} xe^{-x}\,dx &= \lim_{R\to+\infty}\int_0^R xe^{-x}\,dx \\
&= \lim_{R\to+\infty}\left(-\frac{R}{e^R} - \frac{1}{e^R} + 1\right) \\
&= -\lim_{R\to+\infty}\frac{R}{e^R} - \lim_{R\to+\infty}\frac{1}{e^R} + 1 \\
&= -\lim_{R\to+\infty}\frac{R}{e^R} - 0 + 1 \\
&\overset{\text{L'H}}{=} -\lim_{R\to+\infty}\frac{(R)'}{(e^R)'} - 0 + 1 \\
&= -\lim_{R\to+\infty}\frac{1}{e^R} - 0 + 1 \\
&= -0 - 0 + 1 \\
&= \boxed{1}
\end{aligned}$$

(b) For the second integral, we first use the substitution $u = x - 1$, $du = dx$. The limits of integration become $u = 1 - 1 = 0$ and $u = 2 - 1 = 1$. Making the substitutions we get:

$$\int_1^2 \frac{dx}{\sqrt{x-1}} = \int_0^1 \frac{du}{\sqrt{u}}$$

which is a convergent p-integral because $p = \frac{1}{2} < 1$. The value of the integral is:

$$\int_0^1 \frac{du}{\sqrt{u}} = \frac{1}{1 - \frac{1}{2}} = \boxed{2}$$

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Problem 5 Solution

5. Write an integral that represents the length of the curve $y = x^2$ on the interval $0 \leq x \leq 1$. **Do not attempt to evaluate the integral**.

Solution: The arclength is:

$$\begin{aligned} L &= \int_a^b \sqrt{1 + (y')^2}\, dx \\ &= \int_0^1 \sqrt{1 + (2x)^2}\, dx \\ &= \int_0^1 \sqrt{1 + 4x^2}\, dx \end{aligned}$$

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Problem 6 Solution

6. Consider the region bounded by the curves $y = x^2$, $y = 2 - x$, and $x = 0$ for $x \geq 0$.

   (a) Sketch the region.

   (b) Find the volume of the solid obtained by rotating the region about the y-axis.

Solution: We find the volume using the Shell method. The formula we will use is:

$$V = 2\pi \int_a^b x\,(\text{top} - \text{bottom})\, dx$$

where the top curve is $y = 2 - x$ and the bottom curve is $y = x^2$. The lower limit is $a = 0$. The upper limit is the x-coordinate of the point of intersection in the first quadrant. To find the upper limit, we set the y's equal to each other and solve for x.

$$\begin{aligned} y &= y \\ x^2 &= 2 - x \\ x^2 + x - 2 &= 0 \\ (x+2)(x-1) &= 0 \\ x = -2,\ x &= 1 \end{aligned}$$

Therefore, the upper limit of integration is $b = 1$.

The volume is then:

$$\begin{aligned} V &= 2\pi \int_0^1 x\,(\text{top} - \text{bottom})\, dx \\ &= 2\pi \int_0^1 x\left[(2 - x) - (x^2)\right] dx \\ &= 2\pi \int_0^1 \left(2x - x^2 - x^3\right) dx \\ &= 2\pi \left[x^2 - \frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 \\ &= 2\pi \left[1 - \frac{1}{3} - \frac{1}{4}\right] \\ &= \boxed{\frac{5\pi}{6}} \end{aligned}$$

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Problem 7 Solution

7. Determine whether the following series converge or diverge.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n + 2^n}{5^n}$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{2^n}$

Solution:

(a) The first series can be rewritten as the sum of two convergent geometric series:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{3^n + 2^n}{5^n} &= \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n + \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n \\ &= \frac{1}{1-\frac{3}{5}} + \frac{1}{1-\frac{2}{5}} \\ &= \frac{5}{2} + \frac{5}{3} \\ &= \boxed{\frac{25}{6}}\end{aligned}$$

The sum exists so the series converges.

(b) We use the Ratio Test to determine whether or not the second series converges.

$$\begin{aligned}\rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\ &= \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} \\ &= \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n}\right)^2 \\ &= \lim_{n\to\infty} \frac{1}{2}\left(1 + \frac{1}{n}\right)^2 \\ &= \frac{1}{2}\end{aligned}$$

Since $\rho = \frac{1}{2} < 1$, the series $\sum_{n=0}^{+\infty} \frac{n^2}{2^n}$ **converges** by the Ratio Test.