Math 181, Final Exam, Fall 2009
Problem 1 Solution

1. Compute the following indefinite integrals:

(a) $\displaystyle\int x^2 \ln x \, dx$ (b) $\displaystyle\int \frac{x}{x^2+x-6} \, dx$ (c) $\displaystyle\int x \cos x \, dx$

(d) $\displaystyle\int \arctan x \, dx$ (e) $\displaystyle\int e^x \cos x \, dx$

Solution:

(a) We evaluate the integral using Integration by Parts. Let $u = \ln x$ and $v' = x^2$. Then $u' = \dfrac{1}{x}$ and $v = \dfrac{1}{3}x^3$. Using the Integration by Parts formula:

$$\int uv' \, dx = uv - \int u'v \, dx$$

we get:

$$\begin{aligned}
\int x^2 \ln x \, dx &= (\ln x)\left(\frac{1}{3}x^3\right) - \int \frac{1}{x} \cdot \frac{1}{3}x^3 \, dx \\
&= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 \, dx \\
&= \boxed{\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C}
\end{aligned}$$

(b) We compute this integral using Partial Fraction Decomposition. Factoring the denominator and decomposing we get:

$$\frac{x}{x^2+x-6} = \frac{x}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$$

Multiplying the equation by $(x+3)(x-2)$ we get:

$$x = A(x-2) + B(x+3)$$

Next we plug in two different values of x to get a system of two equations in two unknowns (A, B). Letting $x = -3$ and $x = 2$ we get:

$$x = -3: \quad -3 = A(-3-2) + B(-3+3) \quad \Rightarrow \quad A = \frac{3}{5}$$

$$x = 2: \quad 2 = A(2-2) + B(2+3) \quad \Rightarrow \quad B = \frac{2}{5}$$

Plugging these values of A and B back into the decomposed equation and integrating we get:

$$\begin{aligned}\int \frac{x}{x^2+x-6}\,dx &= \int \left(\frac{\frac{3}{5}}{x+3}+\frac{\frac{2}{5}}{x-2}\right)dx \\ &= \boxed{\frac{3}{5}\ln|x+3|+\frac{2}{5}\ln|x-2|+C}\end{aligned}$$

(c) We use Integration by Parts to evaluate the integral. Let $u = x$ and $v' = \cos x$. Then $u' = 1$ and $v = \sin x$. Using the Integration by Parts formula:

$$\int uv'\,dx = uv - \int u'v\,dx$$

we get:

$$\begin{aligned}\int x\cos x\,dx &= x\sin x - \int \sin x\,dx \\ &= \boxed{x\sin x + \cos x}\end{aligned}$$

(d) We will evaluate the first integral using Integration by Parts. Let $u = \arctan x$ and $v' = 1$. Then $u' = \dfrac{1}{x^2+1}$ and $v = x$. Using the Integration by Parts formula:

$$\int uv'\,dx = uv - \int u'v\,dx$$

we get:

$$\int \arctan x\,dx = x\arctan x - \int \frac{1}{x^2+1}\,x\,dx.$$

Use the substitution $w = x^2+1$ to evaluate the integral on the right hand side. Then $dw = 2x\,dx \quad\Rightarrow\quad \frac{1}{2}dw = x\,dx$ and we get:

$$\begin{aligned}\int \arctan x\,dx &= x\arctan x - \frac{1}{2}\int \frac{1}{w}\,dw \\ &= x\arctan x - \frac{1}{2}\ln|w| + C \\ &= \boxed{x\arctan x - \frac{1}{2}\ln(x^2+1)+C}\end{aligned}$$

Note that the absolute value signs aren't needed because $x^2+1>0$ for all x.

(e) $\displaystyle\int e^x\cos x\,dx = \frac{1}{2}e^x\cos x + \frac{1}{2}e^x\sin x + C$

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Problem 2 Solution

2. Compute the definite integral: $\displaystyle\int_0^1 xe^{3x}\,dx$.

Solution: We evaluate the integral using Integration by Parts. Let $u = x$ and $v' = e^{3x}$. Then $u' = 1$ and $v = \frac{1}{3}e^{3x}$. Using the Integration by Parts formula:

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$

we get:

$$\begin{aligned}
\int_0^1 xe^{3x}\,dx &= \left[\frac{1}{3}xe^{3x}\right]_0^1 - \frac{1}{3}\int_0^1 e^{3x}\,dx \\
&= \left[\frac{1}{3}xe^{3x}\right]_0^1 - \frac{1}{3}\left[\frac{1}{3}e^{3x}\right]_0^1 \\
&= \left[\frac{1}{3}(1)e^{3(1)} - \frac{1}{3}(0)e^{3(0)}\right] - \frac{1}{3}\left[\frac{1}{3}e^{3(1)} - \frac{1}{3}e^{3(0)}\right] \\
&= \boxed{\frac{2}{9}e^3 + \frac{1}{9}}
\end{aligned}$$

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Problem 3 Solution

3. Find the 4th midpoint approximation of the integral $\int_0^4 x^2\,dx$ and give your answer in the form $\frac{a}{b}$ where a and b are natural numbers.

Solution: The length of each subinterval of $[0,4]$ is:

$$\Delta x = \frac{b-a}{N} = \frac{4-0}{4} = 1$$

The estimate M_2 is:

$$\begin{aligned} M_2 &= \Delta x \left[f\left(\frac{1}{2}\right) + f\left(\frac{3}{2}\right) + f\left(\frac{5}{2}\right) + f\left(\frac{7}{2}\right) \right] \\ &= 1 \cdot \left[\left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{5}{2}\right)^2 + \left(\frac{7}{2}\right)^2 \right] \\ &= \frac{1}{4} + \frac{9}{4} + \frac{25}{4} + \frac{49}{4} \\ &= \boxed{21} \end{aligned}$$

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Problem 4 Solution

4. Find the volume of the solid that is obtained by revolving the region bounded by the graph of $y = x^2$, the x-axis, $x = 0$ and $x = 1$ around the x-axis ($y = 0$).

Solution: We find the volume using the Disk method. The formula we will use is:

$$V = \pi \int_a^b f(x)^2 \, dx$$

where $a = 0$, $b = 1$, and $f(x) = x^2$. The volume is then:

$$\begin{aligned} V &= \pi \int_0^1 f(x)^2 \, dx \\ &= \pi \int_0^1 (x^2)^2 \, dx \\ &= \pi \int_0^1 x^4 \, dx \\ &= \pi \left[\frac{x^5}{5} \right]_0^1 \\ &= \boxed{\frac{\pi}{5}} \end{aligned}$$

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Problem 5 Solution

5. Compute the integral $\int \frac{dx}{(1-x^2)^{3/2}}$.

Solution: To evaluate the integral we use the trigonometric substitution $x = \sin\theta$. Then $dx = \cos\theta \, d\theta$ and we get:

$$\begin{aligned}
\int \frac{dx}{(1-x^2)^{3/2}} &= \int \frac{\cos\theta \, d\theta}{(1-\sin^2\theta)^{3/2}} \\
&= \int \frac{\cos\theta}{(\cos^2\theta)^{3/2}} \, d\theta \\
&= \int \frac{\cos\theta}{\cos^3\theta} \, d\theta \\
&= \int \frac{1}{\cos^2\theta} \, d\theta \\
&= \int \sec^2\theta \, d\theta \\
&= \tan\theta + C
\end{aligned}$$

Now use the fact that $\sin\theta = x$ and $\cos\theta = \sqrt{1-x^2}$ to write the answer in terms of x.

$$\begin{aligned}
\int \frac{dx}{(1-x^2)^{3/2}} &= \tan\theta + C \\
&= \frac{\sin\theta}{\cos\theta} + C \\
&= \boxed{\frac{x}{\sqrt{1-x^2}} + C}
\end{aligned}$$

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Problem 7 Solution

7. Compute the following two improper integral:

(a) $\displaystyle\int_0^4 \frac{dx}{\sqrt{x}}$ (b) $\displaystyle\int_0^{+\infty} \frac{x\,dx}{x^4+1}$

Solution:

(a) We evaluate the integral as follows

$$\begin{aligned}\int_0^4 \frac{dx}{\sqrt{x}} &= \int_0^1 \frac{dx}{\sqrt{x}} + \int_1^4 \frac{dx}{\sqrt{x}} \\ &= \frac{1}{1-\frac{1}{2}} + \Big[2\sqrt{x}\Big]_1^4 \\ &= 2 + \Big[2\sqrt{4} - 2\sqrt{1}\Big] \\ &= \boxed{4}\end{aligned}$$

We used the fact that $\int_0^1 \frac{dx}{\sqrt{x}}$ is a convergent p-integral.

(b) We evaluate the integral by turning it into a limit calculation.

$$\int_0^{+\infty} \frac{x\,dx}{x^4+1} = \lim_{R\to+\infty} \int_0^R \frac{x\,dx}{x^4+1}$$

To compute the integral we use the u-substitution method with $u = x^2$. Then $\frac{1}{2}\,du = x\,dx$ and we get:

$$\int \frac{x\,dx}{x^4+1} = \frac{1}{2}\int \frac{du}{u^2+1} = \frac{1}{2}\arctan u = \frac{1}{2}\arctan\left(x^2\right)$$

The definite integral from 0 to R is:

$$\begin{aligned}\int_0^R \frac{x\,dx}{x^4+1} &= \left[\frac{1}{2}\arctan\left(x^2\right)\right]_0^R \\ &= \frac{1}{2}\arctan\left(R^2\right) - \frac{1}{2}\arctan\left(0^2\right) \\ &= \frac{1}{2}\arctan\left(R^2\right)\end{aligned}$$

Taking the limit as $R \to +\infty$ we get:

$$\begin{aligned}
\int_0^{+\infty} \frac{x\,dx}{x^4+1} &= \lim_{R\to+\infty} \int_0^R \frac{x\,dx}{x^4+1} \\
&= \lim_{R\to+\infty} \left[\frac{1}{2}\arctan\left(x^2\right)\right] \\
&= \frac{1}{2}\cdot\frac{\pi}{2} \\
&= \boxed{\frac{\pi}{4}}
\end{aligned}$$

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Problem 8 Solution

8. Determine whether the following series converge or not:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

Solution:

(a) We use the Ratio Test to determine whether or not the series converges.

$$\begin{aligned}\rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\ &= \lim_{n\to\infty} \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} \\ &= \lim_{n\to\infty} \frac{1}{3}\left(\frac{n+1}{n}\right)^3 \\ &= \lim_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^3 \\ &= \frac{1}{3}\end{aligned}$$

Since $\rho = \frac{1}{3} < 1$, the series $\sum_{n=1}^{+\infty} \frac{n^3}{3^n}$ **converges** by the Ratio Test.

(b) The series is alternating so we check for absolute convergence by considering the series of absolute values:

$$\sum_{n=1}^{+\infty} \left|\frac{\sin n}{n^2}\right|$$

We note that

$$0 \leq \left|\frac{\sin n}{n^2}\right| \leq \frac{1}{n^2}$$

for $n \geq 1$ and that $\sum_{n=1}^{+\infty} \frac{1}{n^2}$ is a convergent p-series ($p = 2 > 1$). Therefore, the series $\sum_{n=1}^{+\infty} \frac{\sin n}{n^2}$ is absolutely convergent and, thus, **converges**.

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Problem 9 Solution

9. Find the interval of convergence of the power series $\sum_{n=1}^{+\infty} \frac{3^n x^{2n}}{n^2}$. (Remark: Do not forget to examine convergence at the endpoints separately.)

Solution: We use the Ratio Test to find the interval of convergence.

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n\to\infty} \left| \frac{3^{n+1} x^{2(n+1)}}{(n+1)^2} \cdot \frac{n^2}{3^n x^{2n}} \right| \\
&= \lim_{n\to\infty} \left| \frac{3^{n+1}}{3^n} \cdot \frac{n^2}{(n+1)^2} \cdot \frac{x^{2n+2}}{x^{2n}} \right| \\
&= \lim_{n\to\infty} \left| 3 \cdot \left(\frac{n}{n+1} \right)^2 \cdot x^2 \right| \\
&= 3|x|^2 \left(\lim_{n\to\infty} \frac{n}{n+1} \right)^2 \\
&= 3|x|^2 \cdot (1)^2 \\
&= 3|x|^2
\end{aligned}$$

The series converges when $\rho = 3|x|^2 < 1$ which gives us:

$$|x|^2 < \frac{1}{3} \iff |x| < \frac{1}{\sqrt{3}} \iff -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$$

We must now check the endpoints. Plugging $x = \frac{1}{\sqrt{3}}$ into the given power series we get:

$$\sum_{n=1}^{+\infty} \frac{3^n (\frac{1}{\sqrt{3}})^{2n}}{n^2} = \sum_{n=1}^{+\infty} \frac{3^n (\frac{1}{3})^n}{n^2} = \sum_{n=1}^{+\infty} \frac{1}{n^2}$$

which is a convergent p-series ($p = 2 > 1$). Plugging in $x = -\frac{1}{\sqrt{3}}$ we get:

$$\sum_{n=1}^{+\infty} \frac{3^n (-\frac{1}{\sqrt{3}})^{2n}}{n^2} = \sum_{n=1}^{+\infty} \frac{3^n (\frac{1}{3})^n}{n^2} = \sum_{n=1}^{+\infty} \frac{1}{n^2}$$

which, again, is a convergent p-series ($p = 2 > 1$). Thus, the interval of convergence is:

$$\boxed{-\frac{1}{\sqrt{3}} \leq x \leq \frac{1}{\sqrt{3}}}$$