Math 181, Final Exam, Fall 2014
Problem 1 Solution

1. (a) Evaluate the integral $\int \ln\left(\sqrt{x}\right)\,dx$.

Solution: We use a property of logarithms to rewrite the integral as

$$\int \ln\left(\sqrt{x}\right)\,dx = \frac{1}{2}\int \ln(x)\,dx.$$

Letting $u = \ln(x)$ and $dv = dx$ yields $du = \frac{1}{x}\,dx$ and $v = x$. Integration by parts yields the result:

$$\begin{aligned}
\int \ln\left(\sqrt{x}\right)\,dx &= \frac{1}{2}\ln(x)\,dx \\
\int \ln\left(\sqrt{x}\right)\,dx &= \frac{1}{2}\left(x\ln(x) - \int x\cdot\frac{1}{x}\,dx\right) \\
\int \ln\left(\sqrt{x}\right)\,dx &= \frac{1}{2}\left(x\ln(x) - \int dx\right) \\
\int \ln\left(\sqrt{x}\right)\,dx &= \frac{1}{2}x\ln(x) - \frac{1}{2}x + C
\end{aligned}$$

1. (b) Evaluate the integral $\int \frac{x^2+2x}{x^2-1}\,dx$.

Solution: The rational function is improper (the degree of the numerator is $\geq$ the degree of the denominator). Thus, we use long division.

$$\begin{array}{r|l}
 & \quad\quad\quad\quad\quad 1 \\ \hline
x^2-1 & \quad x^2+2x \\
 & -x^2 \quad\quad +1 \\ \hline
 & \quad\quad\quad 2x+1
\end{array}$$

The integral may be rewritten now as

$$\int \frac{x^2+2x}{x^2-1}\,dx = \int \left(1 + \frac{2x+1}{x^2-1}\right)\,dx = x + \int \frac{2x+1}{x^2-1}\,dx.$$

The integrand on the right hand side may be decomposed as follows:

$$\frac{2x+1}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1}.$$

Clearing denominators yields $2x+1 = A(x-1) + B(x+1)$. When $x = 1$ we have $B = \frac{3}{2}$. When $x = -1$ we have $A = \frac{1}{2}$. Thus, the integral becomes:

$$\int \frac{x^2+2x}{x^2-1}\,dx = x + \int \left(\frac{\frac{1}{2}}{x+1} + \frac{\frac{3}{2}}{x-1}\right) = x + \frac{1}{2}|x+1| + \frac{3}{2}\ln|x-1| + C.$$

1. (c) Evaluate the integral $\int \frac{dx}{4x^2+1}$.

Solution: Let $u = 2x$. Then $\frac{1}{2}\,du = dx$ so that

$$\begin{aligned}\int \frac{dx}{4x^2+1} &= \int \frac{dx}{(2x)^2+1} \\ &= \int \frac{\frac{1}{2}\,du}{u^2+1} \\ &= \frac{1}{2}\int \frac{du}{u^2+1} \\ &= \frac{1}{2}\arctan(u) + C \\ &= \frac{1}{2}\arctan(2x) + C\end{aligned}$$

1. (d) Evaluate the integral $\int \sin^3(x)\,dx$.

Solution: We begin by rewriting the function as follows:

$$\sin^3(x) = \sin(x)\sin^2(x) = \sin(x)(1-\cos^2(x))$$

Then we let $u = \cos(x)$ so that $-du = \sin(x)\,dx$ and we get

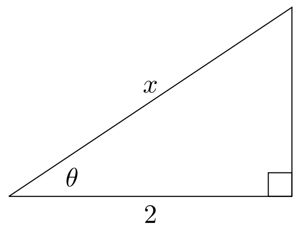
$$\begin{aligned}\int \sin^3(x)\,dx &= \int \sin(x)(1-\cos^2(x))\,dx \\ &= -\int (1-u^2)\,du \\ &= -u + \frac{u^3}{3} + C \\ &= -\cos(x) + \frac{1}{3}\cos^3(x) + C\end{aligned}$$

1. (e) Evaluate the integral $\int \frac{dx}{\sqrt{x^2-4}}$.

Solution: We begin by letting $x = 2\sec\theta$ so that $dx = 2\sec\theta\tan\theta\,d\theta$. After substituting into the integral we obtain:

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2-4}} &= \int \frac{2\sec\theta\tan\theta\,d\theta}{\sqrt{(2\sec\theta)^2-4}} \\ &= \int \frac{2\sec\theta\tan\theta}{\sqrt{4\sec^2\theta-4}}\,d\theta \\ &= \int \frac{2\sec\theta\tan\theta}{\sqrt{4\tan^2\theta}}\,d\theta \\ &= \int \frac{2\sec\theta\tan\theta}{2\tan\theta}\,d\theta \\ &= \int \sec\theta\,d\theta \\ &= \ln|\sec\theta+\tan\theta| + C\end{aligned}$$

Since $x = 2\sec\theta$ and, thus, $\sec\theta = \frac{x}{2} = \frac{\text{hypotenuse}}{\text{adjacent}}$ we can construct a right triangle with hypotenuse x and adjacent side 2.



From the above figure we obtain $\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{x^2-4}}{2}$ where the opposite side was obtain using the Pythageorean Theorem. Therefore, the final answer is

$$\int \frac{dx}{\sqrt{x^2-4}} = \ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C$$

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Problem 2 Solution

2. (a) Determine whether the integral converges or diverges. If it converges, compute the integral.

$$\int_e^\infty \frac{1}{x\ln(x)}\,dx$$

Solution: The integral is improper so we transform it into a limit calculation:

$$\int_e^\infty \frac{1}{x\ln(x)}\,dx = \lim_{b\to\infty}\int_e^b \frac{1}{x\ln(x)}\,dx$$

Let $u = \ln(x)$ so that $du = \frac{1}{x}\,dx$. Then

$$\begin{aligned}\int_e^\infty \frac{1}{x\ln(x)}\,dx &= \lim_{b\to\infty}\int_e^b \frac{1}{x\ln(x)}\,dx \\ &= \lim_{b\to\infty}\int_1^{\ln(b)} \frac{1}{u}\,du \\ &= \lim_{b\to\infty} \ln(\ln(b)) \\ &= +\infty\end{aligned}$$

Thus, the integral diverges.

2. (b) Determine whether the integral converges or diverges. If it converges, compute the integral.

$$\int_0^\infty \frac{x}{x^4+1}\,dx$$

Solution: The integral is improper so we transform it into a limit calculation:

$$\int_0^\infty \frac{x}{x^4+1}\,dx = \lim_{b\to\infty}\int_0^b \frac{x}{x^4+1}\,dx$$

Let $u = x^2$ so that $\frac{1}{2}\,du = x\,dx$. Then

$$\begin{aligned}\int_0^\infty \frac{x}{x^4+1}\,dx &= \lim_{b\to\infty}\int_0^b \frac{x}{x^4+1}\,dx \\ &= \frac{1}{2}\lim_{b\to\infty}\int_0^{b^2} \frac{1}{u^2+1}\,du \\ &= \frac{1}{2}\lim_{b\to\infty} \arctan(b^2) \\ &= \frac{1}{2}\cdot\frac{\pi}{2} \\ &= \frac{\pi}{4}\end{aligned}$$

Thus, the integral converges and its value is $\frac{\pi}{4}$.

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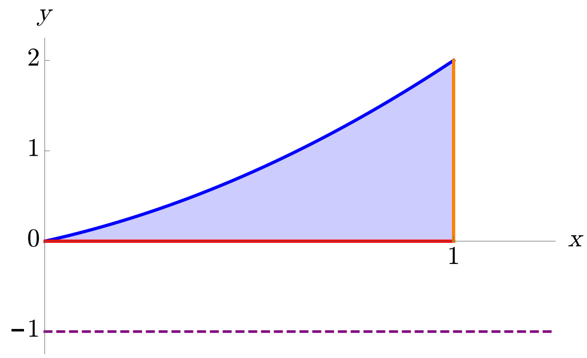
Problem 3 Solution

3. Let $\mathcal{R}$ be the region bounded by the curves $y = 0$ and $y = x^2 + x$ between $x = 0$ and $x = 1$. Compute the volume of the solid of revolution obtained when $\mathcal{R}$ is rotated about the axis $y = -1$.

Solution: A plot of the region and the axis is shown below.



We use the slicing method to find the volume. The slices are washers and the inner and outer radii are:

$$r_i = 1, \quad r_o = x^2 + x + 1$$

noting that the 1 present in both equations is the distance between the x-axis and the axis $y = -1$. Therefore, the volume is:

$$\begin{aligned}
V &= \pi \int_a^b (r_o^2 - r_i^2)\,dx \\
&= \pi \int_0^1 \left[(x^2 + x + 1)^2 - 1^2\right]\,dx \\
&= \pi \int_0^1 \left(x^4 + 2x^3 + 3x^2 + 2x\right)\,dx \\
&= \pi \left[\frac{x^5}{5} + \frac{x^4}{2} + x^3 + x^2\right]_0^1 \\
&= \pi \left[\frac{1}{5} + \frac{1}{2} + 1 + 1\right] \\
&= \frac{27\pi}{10}
\end{aligned}$$

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Problem 4 Solution

4. (a) Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{2^n}{n^3}$$

Solution: Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2^n}{n^3} = +\infty \neq 0$ the series diverges by the Divergence Test.

4. (b) Determine whether the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$$

Solution: The series is alternating. The function $a_n = \frac{1}{\ln(n)}$ is decreasing and approaches 0 as $n \to \infty$. Therefore, the series converges by the Alternating Series Test.

4. (c) Determine whether the following series converges or diverges.

$$\sum_{n=5}^{\infty} \frac{1}{\sqrt{n}-2}$$

Solution: Consider the series

$$\sum a_n = \sum_{n=5}^{\infty} \frac{1}{\sqrt{n}-2} \quad \text{and} \quad \sum b_n = \sum_{n=5}^{\infty} \frac{1}{\sqrt{n}}$$

Both series contain positive terms and we know that $\sum b_n$ is a divergent p-series (because $p = \frac{1}{2} \leq 1$). Furthermore,

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}-2} = 1$$

Thus, since $0 < L < \infty$ and $\sum b_n$ diverges the series $\sum a_n$ diverges by the Limit Comparison Test.

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Problem 5 Solution

5. (a) Find the third Taylor polynomial, $p_3(x)$, centered at $a = 0$ for the function

$$f(x) = \sqrt{x+1}$$

Solution: The function and its first three derivatives evaluated at 0 are:

$$\begin{aligned} f(x) &= (1+x)^{1/2} & f(0) &= 1 \\ f'(x) &= \frac{1}{2}(1+x)^{-1/2} & f'(0) &= \frac{1}{2} \\ f''(x) &= -\frac{1}{4}(1+x)^{-3/2} & f''(0) &= -\frac{1}{4} \\ f'''(x) &= \frac{3}{8}(1+x)^{-5/2} & f'''(0) &= \frac{3}{8} \end{aligned}$$

Therefore, the third order Taylor polynomial is

$$\begin{aligned} p_3(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \\ p_3(x) &= 1 + \frac{1}{2}x - \frac{1}{4 \cdot 2!}x^2 + \frac{3}{8 \cdot 3!}x^3 \\ p_3(x) &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 \end{aligned}$$

5. (b) Find the third Taylor polynomial, $p_3(x)$, centered at $a = 0$ for the function

$$f(x) = \frac{1}{1-3x^2}$$

Solution: We use the fact that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

and replace x with $3x^2$ to obtain

$$\begin{aligned} \frac{1}{1-3x^2} &= 1 + (3x^2) + (3x^2)^2 + (3x^2)^3 + \cdots \\ &= 1 + 3x^2 + 9x^4 + 27x^6 + \cdots \end{aligned}$$

Thus, the third order Taylor polynomial is

$$p_3(x) = 1 + 3x^2$$

5. (c) Find the third Taylor polynomial, $p_3(x)$, centered at $a = 0$ for the function

$$f(x) = xe^x$$

Solution: We use the fact that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

and multiply by x to obtain

$$\begin{aligned} xe^x &= x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) \\ &= x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots \end{aligned}$$

Thus, the third order Taylor polynomial is

$$p_3(x) = x + x^2 + \frac{x^3}{2!}$$

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Problem 6 Solution

6. The function $f(x)$ is defined by the power series:

$$f(x) = \sum_{n=1}^{\infty} n5^n x^n$$

(a) Find the radius of convergence of the power series.

(b) Find $f''(0)$.

(c) Does the series converge at $x = -\frac{1}{5}$?

Solution:

(a) We use the Ratio Test to determine when the series converges absolutely.

$$\begin{aligned} r &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n\to\infty} \left| \frac{(n+1)5^{n+1}x^{n+1}}{n5^n x^n} \right| \\ &= 5|x| \lim_{n\to\infty} \frac{n+1}{n} \\ &= 5|x| \end{aligned}$$

We know that, by the Ratio Test, the series converges if $r < 1$. Thus, we need $5|x| < 1$ or $|x| < \frac{1}{5}$. Therefore, the radius of convergence is $\frac{1}{5}$.

(b) The series can be expanded as follows:

$$\begin{aligned} f(x) = \sum_{n=1}^{\infty} n5^n x^n &= 5x + 2\cdot 5^2 x^2 + 3 \cdot 5^3 x^3 + \cdots \\ &= 5x + 50x^2 + 375x^3 + \cdots \end{aligned}$$

The first and second derivatives of f are:

$$\begin{aligned} f'(x) &= 5 + 100x + 1125x^2 + \cdots \\ f''(x) &= 100 + 2250x + \cdots \end{aligned}$$

When evaluated at $x = 0$ we have

$$f''(0) = 100 + 2250 \cdot 0 + \cdots = 100$$

because the higher order terms in the series will evaluate to 0.

(c) When $x = -\frac{1}{5}$ we have

$$\sum_{n=1}^{\infty} n5^n x^n = \sum_{n=1}^{\infty} n5^n \left(-\frac{1}{5}\right)^n = \sum_{n=1}^{\infty} n(-1)^n$$

The series diverges by the Divergence Test because

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n(-1)^n \text{ does not exist}$$

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Problem 7 Solution

7. The equation of a curve $\mathcal{C}$ is given in polar coordinates as

$$r = 2\cos\theta \quad \text{for} \;\; 0 \leq \theta \leq 2\pi$$

(a) Rewrite the equation in Cartesian coordinates.

(b) Find the Cartesian coordinates (x, y) of all the points on the curve at which the tangent to the curve is horizontal.

Solution:

(a) We begin by multiplying both sides of the given equation by r to obtain

$$r^2 = 2r\cos\theta$$

We then use the fact that $x^2 + y^2 = r^2$ and $x = r\cos\theta$ to obtain the equation

$$x^2 + y^2 = 2x$$

While this is perfectly sufficient as an answer, it would be best to further rewrite the equation in order to solve part (b). Subtracting $2x$ from both sides yields the result

$$x^2 - 2x + y^2 = 0$$

Then completing the square on the x terms gives us

$$(x-1)^2 + y^2 = 1$$

We see that this is a circle of radius 1 centered at $(1, 0)$.

(b) The horizontal tangent lines to the circle will occur at the points $(1, 1)$ and $(1, -1)$.