Math 181, Final Exam, Spring 2010
Problem 1 Solution

1. Integrate: (a) $\displaystyle\int x\cos(3x)\,dx$ (b) $\displaystyle\int \frac{\sin x}{\sqrt{3-\cos x}}\,dx$

Solution:

(a) We will evaluate the integral using Integration by Parts. Let $u = x$ and $v' = \cos(3x)$. Then $u' = 1$ and $v = \frac{1}{3}\sin(3x)$. Using the Integration by Parts formula:

$$\int uv'\,dx = uv - \int u'v\,dx$$

we get:

$$\begin{aligned}
\int x\cos(3x)\,dx &= \frac{1}{3}x\sin(3x) - \int \frac{1}{3}\sin(3x)\,dx \\
&= \frac{1}{3}x\sin(3x) - \left[-\frac{1}{9}\cos(3x)\right] + C \\
&= \boxed{\frac{1}{3}x\sin(3x) + \frac{1}{9}\cos(3x) + C}
\end{aligned}$$

(b) We will evaluate the integral using the u-substitution method. Let $u = 3 - \cos x$. Then $du = \sin x\,dx$ and we get:

$$\begin{aligned}
\int \frac{\sin x}{\sqrt{3-\cos x}}\,dx &= \int \frac{1}{\sqrt{u}}\,du \\
&= \int u^{-1/2}\,du \\
&= 2u^{1/2} + C \\
&= \boxed{2\sqrt{3-\cos x} + C}
\end{aligned}$$

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Problem 2 Solution

2. Evaluate: (a) $\displaystyle\int_0^1 \frac{dx}{4-x^2}$ (b) $\displaystyle\int_1^e x^2 \ln x \, dx$

Solution:

(a) We will evaluate the integral using Partial Fraction Decomposition. First, we factor the denominator and then decompose the rational function into a sum of simpler rational functions.

$$\frac{1}{4-x^2} = \frac{1}{(2-x)(2+x)} = \frac{A}{2-x} + \frac{B}{2+x}$$

Next, we multiply the above equation by $(2-x)(2+x)$ to get:

$$1 = A(2+x) + B(2-x)$$

Then we plug in two different values for x to create a system of two equations in two unknowns (A, B). We select $x = 2$ and $x = -2$ for simplicity.

$$x = 2: \; A(2+2) + B(2-2) = 1 \quad \Rightarrow \quad A = \frac{1}{4}$$
$$x = -2: \; A(2-2) + B(2+2) = 1 \quad \Rightarrow \quad B = \frac{1}{4}$$

Finally, we plug these values for A and B back into the decomposition and integrate.

$$\begin{aligned}
\int_0^1 \frac{1}{4-x^2}\,dx &= \int_0^1 \left(\frac{A}{2-x} + \frac{B}{2+x}\right) dx \\
&= \int_0^1 \left(\frac{\frac{1}{4}}{2-x} + \frac{\frac{1}{4}}{2+x}\right) dx \\
&= \left[-\frac{1}{4}\ln|2-x| + \frac{1}{4}\ln|2+x|\right]_0^1 \\
&= \left[-\frac{1}{4}\ln|2-1| + \frac{1}{4}\ln|2+1|\right] - \left[-\frac{1}{4}\ln|2-0| + \frac{1}{4}\ln|2+0|\right] \\
&= \boxed{\frac{1}{4}\ln 3}
\end{aligned}$$

(b) We evaluate the integral using Integration by Parts. Let $u = \ln x$ and $v' = x^2$. Then $u' = \dfrac{1}{x}$ and $v = \dfrac{1}{3}x^3$. Using the Integration by Parts formula:

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$

we get:

$$\begin{aligned}
\int_1^e x^2 \ln x \, dx &= \left[(\ln x) \left(\frac{1}{3} x^3 \right) \right]_1^e - \int_1^e \frac{1}{x} \cdot \frac{1}{3} x^3 \, dx \\
&= \left[\frac{1}{3} x^3 \ln x \right]_1^e - \frac{1}{3} \int_1^e x^2 \, dx \\
&= \left[\frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 \right]_1^e \\
&= \left[\frac{1}{3} e^3 \ln e - \frac{1}{9} e^3 \right] - \left[\frac{1}{3} (1)^3 \ln 1 - \frac{1}{9} (1)^3 \right] \\
&= \boxed{\frac{2}{9} e^3 + \frac{1}{9}}
\end{aligned}$$

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Problem 3 Solution

3. Determine whether each series converges or diverges. If the series converges, determine whether the convergence is absolute or conditional.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{2^n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n+1}}{n^2}$ (c) $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{\sqrt{n^2+1}}$

Solution:

(a) We use the Ratio Test to determine whether or not the series converges.

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\
&= \lim_{n\to\infty} \frac{(n+1)^3}{2^{n+1}} \cdot \frac{2^n}{n^3} \\
&= \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n}\right)^3 \\
&= \lim_{n\to\infty} \frac{1}{2}\left(1+\frac{1}{n}\right)^3 \\
&= \frac{1}{2}
\end{aligned}$$

Since $\rho = \frac{1}{2} < 1$, the series $\sum_{n=1}^{+\infty} \frac{n^3}{2^n}$ **converges** by the Ratio Test.

(b) The series is alternating so we check for absolute convergence by considering the series of absolute values:

$$\sum_{n=1}^{+\infty} \left|\frac{(-1)^n\sqrt{n+1}}{n^2}\right| = \sum_{n=1}^{+\infty} \frac{\sqrt{n+1}}{n^2}$$

We note that:

$$0 \le \frac{\sqrt{n+1}}{n^2} \le \frac{\sqrt{n+n}}{n^2} = \frac{\sqrt{2}}{n^{3/2}}$$

for $n \ge 1$ and that $\sum_{n=1}^{+\infty} \frac{\sqrt{2}}{n^{3/2}}$ is a convergent p-series with $p = \frac{3}{2} > 1$. Therefore, the series $\sum_{n=1}^{+\infty} \frac{\sqrt{n+1}}{n^2}$ converges by the Comparison Test and $\sum_{n=1}^{+\infty} \frac{(-1)^n\sqrt{n+1}}{n^2}$ is **absolutely convergent**.

(c) We use the Limit Comparison Test with $\sum_{n=1}^{+\infty} \frac{1}{n}$ which is a divergent p-series with $p = 1 \le 1$.

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{a_n}{b_n} \\
&= \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} \\
&= 1
\end{aligned}$$

Since $L = 1$ and $\sum_{n=1}^{+\infty} \frac{1}{n}$ diverges, the series $\sum_{n=1}^{+\infty} \frac{1}{\sqrt{n^2+1}}$ **diverges**.

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Problem 4 Solution

4. Determine whether the improper integrals converge or not (justify your answers):

(a) $\displaystyle\int_1^{+\infty} \frac{dx}{x^2+x+1}$ (b) $\displaystyle\int_0^{\pi/2} \tan x\, dx$

Solution:

(a) We will use the Comparison Test to show that the integral converges. Let $g(x) = \frac{1}{x^2+x+1}$. We must choose a function $f(x)$ that satisfies:

$$(1)\ \int_1^{+\infty} f(x)\,dx \text{ converges} \quad \text{and} \quad (2)\ 0 \le g(x) \le f(x) \text{ for } x \ge 1$$

We choose $f(x) = \frac{1}{x^2}$. This function satisfies the inequality:

$$0 \le g(x) \le f(x)$$
$$0 \le \frac{1}{x^2+x+1} \le \frac{1}{x^2}$$

for $x \ge 1$ because the denominator of $g(x)$ is greater than the denominator of $f(x)$ for these values of x. Furthermore, the integral $\int_1^{+\infty} f(x)\,dx = \int_1^{+\infty} \frac{1}{x^2}\,dx$ converges because it is a p-integral with $p = 2 > 1$. Therefore, the integral $\int_1^{+\infty} g(x)\,dx = \int_1^{+\infty} \frac{1}{x^2+x+1}\,dx$ converges by the Comparison Test.

(b) We begin by noting that $\tan x$ is undefined at $x = \frac{\pi}{2}$. Thus, we replace the upper limit with R and take the limit as $R \to \frac{\pi}{2}$.

$$\begin{aligned}
\int_0^{\pi/2} \tan x\,dx &= \lim_{R\to\frac{\pi}{2}} \int_0^R \tan x\,dx \\
&= \lim_{R\to\frac{\pi}{2}} \Big[-\ln|\cos x|\Big]_0^R \\
&= \lim_{R\to\frac{\pi}{2}} \Big[-\ln|\cos R| + \ln|\cos 0|\Big] \\
&= \infty + 0 \\
&= \infty
\end{aligned}$$

Therefore, the integral diverges.

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Problem 5 Solution

5. Compute the arclength of the graph of $y = (x+1)^{3/2} + 1$ between $x = 0$ and $x = 2$.

Solution: The arclength is:

$$\begin{aligned} L &= \int_a^b \sqrt{1 + (y')^2}\, dx \\ &= \int_0^2 \sqrt{1 + \left(\frac{3}{2}(x+1)^{1/2}\right)^2}\, dx \\ &= \int_0^2 \sqrt{1 + \frac{9}{4}(x+1)}\, dx \\ &= \int_0^2 \sqrt{\frac{9}{4}x + \frac{13}{4}}\, dx \end{aligned}$$

We now use the u-substitution $u = \frac{9}{4}x + \frac{13}{4}$. Then $\frac{4}{9}\, du = dx$, the lower limit of integration changes from 0 to $\frac{13}{4}$, and the upper limit of integration changes from 2 to $\frac{31}{4}$.

$$\begin{aligned} L &= \int_0^2 \sqrt{\frac{9}{4}x + \frac{13}{4}}\, dx \\ &= \frac{4}{9}\int_{13/4}^{31/4} \sqrt{u}\, du \\ &= \frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_{13/4}^{31/4} \\ &= \frac{4}{9}\left[\frac{2}{3}\left(\frac{31}{4}\right)^{3/2} - \frac{2}{3}\left(\frac{13}{4}\right)^{3/2}\right] \\ &= \boxed{\frac{8}{27}\left[\left(\frac{31}{4}\right)^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right]} \end{aligned}$$

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Problem 6 Solution

6. Find the volume of the solid that is obtained by revolving the region below the graph $y = x^2 - 1$ about the x-axis for $1 \leq x \leq 2$.

Solution: We find the volume using the Disk method. The formula we will use is:

$$V = \pi \int_a^b f(x)^2 \, dx$$

where $a = 1$, $b = 2$, and $f(x) = x^2 - 1$. The volume is then:

$$\begin{aligned}
V &= \pi \int_1^2 f(x)^2 \, dx \\
&= \pi \int_1^2 (x^2 - 1)^2 \, dx \\
&= \pi \int_1^2 (x^4 - 2x^2 + 1) \, dx \\
&= \pi \left[\frac{x^5}{5} - \frac{2x^3}{3} + x \right]_1^2 \\
&= \pi \left[\left(\frac{32}{5} - \frac{16}{3} + 2 \right) - \left(\frac{1}{5} - \frac{2}{3} + 1 \right) \right] \\
&= \boxed{\frac{38\pi}{15}}
\end{aligned}$$

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Problem 7 Solution

7. Find the Maclaurin series around $x = 0$ for $f(x) = \ln(1 + 2x)$.

Solution: We begin by recalling the Maclaurin series for $\dfrac{1}{1-x}$:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n \quad \text{for} \quad -1 < x < 1.$$

Upon replacing x with $-2x$ we find that:

$$\frac{1}{1+2x} = 1 + (-2x) + (-2x)^2 + \cdots = \sum_{n=0}^{\infty} (-2x)^n = \sum_{n=0}^{\infty} (-2)^n x^n \quad \text{for} \quad -\frac{1}{2} < x < \frac{1}{2}$$

Since

$$\int \frac{1}{1+2x}\, dx = \frac{1}{2}\ln(1+2x)$$

we have the relation

$$2\int \frac{1}{1+2x}\, dx = \ln(1+2x)$$

provided that $x > -\frac{1}{2}$. The above relation yields the Maclaurin series for $\ln(1 + 2x)$ on the interval $-\frac{1}{2} < x < \frac{1}{2}$ as follows:

$$\ln(1+2x) = 2\int \frac{1}{1+2x}\, dx = 2\int \sum_{n=0}^{\infty} (-2)^n x^n\, dx = 2\sum_{n=0}^{\infty} \frac{(-2)^n}{n+1} x^{n+1}.$$

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Problem 8 Solution

8. Find the interval of convergence for $\sum_{n=3}^{\infty} \frac{(2x)^n}{\ln n}$.

Solution: We use the Ratio Test to find the interval of convergence.

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n\to\infty} \left| \frac{(2x)^{n+1}}{\ln(n+1)} \cdot \frac{\ln n}{(2x)^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{\ln n}{\ln(n+1)} \cdot \frac{2^{n+1}}{2^n} \cdot \frac{x^{n+1}}{x^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{\ln n}{\ln(n+1)} \cdot 2 \cdot x \right| \\
&= 2|x| \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} \\
&= 2|x| \cdot (1) \\
&= 2|x|
\end{aligned}$$

The series converges when $\rho = 2|x| < 1$ which gives us:

$$|x| < \frac{1}{2} \quad \Longleftrightarrow \quad -\frac{1}{2} < x < \frac{1}{2}$$

We must now check the endpoints. Plugging $x = \frac{1}{2}$ into the given power series we get:

$$\sum_{n=1}^{+\infty} \frac{(2(\frac{1}{2}))^n}{\ln n} = \sum_{n=1}^{+\infty} \frac{1}{\ln n}$$

This series diverges by a direct comparison with $\sum_{n=1}^{+\infty} \frac{1}{n}$ which is divergent.

Plugging in $x = -\frac{1}{2}$ we get:

$$\sum_{n=1}^{+\infty} \frac{(2(-\frac{1}{2}))^n}{\ln n} = \sum_{n=1}^{+\infty} \frac{(-1)^n}{\ln n}$$

which converges by the Leibniz Test. Thus, the interval of convergence is:

$$\boxed{-\frac{1}{2} \le x < \frac{1}{2}}$$