Math 210, Exam 1, Fall 2005
Problem 1 Solution

1. Consider the triangle with vertices

$$A = (1, -3, -2), \quad B = (2, 0, -4), \quad C = (6, -2, -5).$$

(a) Find the area of this triangle.

(b) Determine whether or not it is a right triangle.

Solution:

(a) The area of the triangle is half the magnitude of the cross product of $\overrightarrow{AB} = \langle 1, 3, -2 \rangle$ and $\overrightarrow{BC} = \langle 4, -2, -1 \rangle$, which represents the area of the parallelogram spanned by the two vectors. The cross product of these two vector is computed as follows:

$$\vec{\mathbf{n}} = \overrightarrow{AB} \times \overrightarrow{BC}$$

$$\vec{\mathbf{n}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 3 & -2 \\ 4 & -2 & -1 \end{vmatrix}$$

$$\vec{\mathbf{n}} = \hat{\mathbf{i}} \begin{vmatrix} 3 & -2 \\ -2 & -1 \end{vmatrix} - \hat{\mathbf{j}} \begin{vmatrix} 1 & -2 \\ 4 & -1 \end{vmatrix} + \hat{\mathbf{k}} \begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix}$$

$$\vec{\mathbf{n}} = \hat{\mathbf{i}}\,[(3)(-1) - (-2)(-2)] - \hat{\mathbf{j}}\,[(1)(-1) - (4)(-2)] + \hat{\mathbf{k}}\,[(1)(-2) - (4)(3)]$$

$$\vec{\mathbf{n}} = -7\,\hat{\mathbf{i}} - 7\,\hat{\mathbf{j}} - 14\,\hat{\mathbf{k}}$$

$$\vec{\mathbf{n}} = \langle -7, -7, -14 \rangle$$

Thus, the area of the triangle is:

$$A = \frac{1}{2} \left\| \overrightarrow{AB} \times \overrightarrow{BC} \right\|$$

$$A = \frac{1}{2}\sqrt{(-7)^2 + (-7)^2 + (-14)^2}$$

$$A = \frac{1}{2}\sqrt{294}$$

$$\boxed{A = \frac{7\sqrt{6}}{2}}$$

(b) We note that the dot product of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ is:

$$\overrightarrow{AB} \cdot \overrightarrow{BC} = \langle 1, 3, -2 \rangle \cdot \langle 4, -2, -1 \rangle$$

$$\overrightarrow{AB} \cdot \overrightarrow{BC} = (1)(4) + (3)(-2) + (-2)(-1)$$

$$\boxed{\overrightarrow{AB} \cdot \overrightarrow{BC} = 0}$$

Since the dot product is 0, we know that the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are orthogonal. Thus, triangle ABC is a right triangle.

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Problem 2 Solution

2. Find an equation for the plane which contains the point $(2, -1, 5)$ and the line

$$\frac{x+1}{4} = \frac{y-4}{2} = z - 1$$

Solution: To find an equation for the plane we need two more points that lie in the plane, which we obtain from the equations for the line. The point $(x, y, z) = (-1, 4, 1)$ is on the line because

$$\frac{-1+1}{4} = \frac{4-4}{2} = 1 - 1 = 0$$

The point $(x, y, z) = (3, 6, 2)$ is also on the line because

$$\frac{3+1}{4} = \frac{6-4}{2} = 2 - 1 = 1$$

Let $A = (2, -1, 5)$, $B = (-1, 4, 1)$, and $C = (3, 6, 2)$. In order to find an equation for the plane containing A, B, and C we need a vector $\overrightarrow{\mathbf{n}}$ perpendicular to it. We let $\overrightarrow{\mathbf{n}}$ be the cross product of $\overrightarrow{AB} = \langle -3, 5, -4 \rangle$ and $\overrightarrow{BC} = \langle 4, 2, 1 \rangle$ because these vectors lie in the plane.

$$\begin{aligned}
\overrightarrow{\mathbf{n}} &= \overrightarrow{AB} \times \overrightarrow{BC} \\
\overrightarrow{\mathbf{n}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -3 & 5 & -4 \\ 4 & 2 & 1 \end{vmatrix} \\
\overrightarrow{\mathbf{n}} &= \hat{\mathbf{i}} \begin{vmatrix} 5 & -4 \\ 2 & 1 \end{vmatrix} - \hat{\mathbf{j}} \begin{vmatrix} -3 & -4 \\ 4 & 1 \end{vmatrix} + \hat{\mathbf{k}} \begin{vmatrix} -3 & 5 \\ 4 & 2 \end{vmatrix} \\
\overrightarrow{\mathbf{n}} &= \hat{\mathbf{i}}\,[(5)(1) - (2)(-4)] - \hat{\mathbf{j}}\,[(-3)(1) - (4)(-4)] + \hat{\mathbf{k}}\,[(-3)(2) - (4)(5)] \\
\overrightarrow{\mathbf{n}} &= 13\,\hat{\mathbf{i}} - 13\,\hat{\mathbf{j}} - 26\,\hat{\mathbf{k}} \\
\overrightarrow{\mathbf{n}} &= \langle 13, -13, -26 \rangle
\end{aligned}$$

Using $A = (2, -1, 5)$ as a point in the plane, we have:

$$\boxed{13(x-2) - 13(y+1) - 26(z-5) = 0}$$

as an equation for the plane containing the given point and line.

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Problem 3 Solution

3. For the position function $\overrightarrow{\mathbf{r}}(t) = \langle t, t^2, t^3 \rangle$, find the velocity $\overrightarrow{\mathbf{v}}(t)$, the speed $v(t)$, and the acceleration $\overrightarrow{\mathbf{a}}(t)$.

Solution: The velocity, acceleration, and speed functions are:

$$\begin{aligned}
\overrightarrow{\mathbf{v}}(t) &= \overrightarrow{\mathbf{r}}'(t) = \left\langle 1, 2t, 3t^2 \right\rangle \\
\overrightarrow{\mathbf{a}}(t) &= \overrightarrow{\mathbf{v}}'(t) = \langle 0, 2, 6t \rangle \\
v(t) &= \left|\left|\overrightarrow{\mathbf{v}}(t)\right|\right| \\
&= \sqrt{1^2 + (2t)^2 + (3t^2)^2} \\
&= \sqrt{1 + 4t^2 + 9t^4}
\end{aligned}$$

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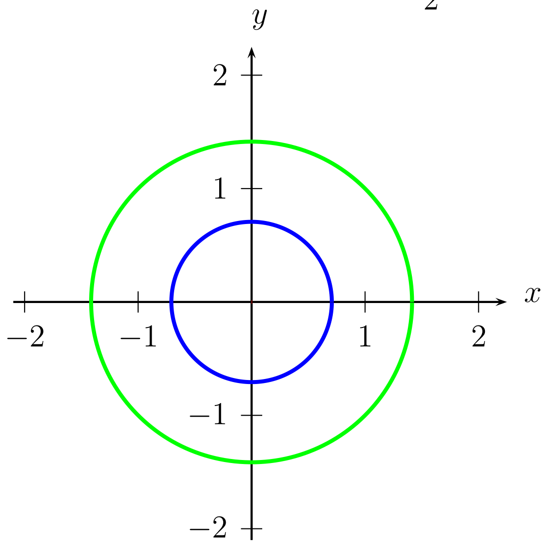
Problem 4 Solution

4. Sketch the level sets for the function $f(x, y) = 4x^2 + 4y^2 + 2$ which correspond to the function values 2, 4, and 10.

Solution: The level sets of $f(x, y) = 4x^2 + 4y^2 + 2$ are the curves obtained by setting $f(x, y)$ to a constant C.

$$C = 4x^2 + 4y^2 + 2 \quad \Longleftrightarrow \quad x^2 + y^2 = \frac{C-2}{4}$$

These curves are circles centered at $(0, 0)$ with radius $\frac{\sqrt{C-2}}{2}$.



Note that $C = 10$ is the green circle with radius $\sqrt{2}$, $C = 4$ is the blue circle with radius $\frac{\sqrt{2}}{2}$, and $C = 0$ is the origin (because the radius is 0).

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Problem 5 Solution

5. Evaluate the following limit, or show it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

Solution: The function $f(x,y) = \dfrac{xy}{x^2+y^2}$ is not continuous at $(0,0)$ as the point is not in the domain of f. If the limit exists, the value of the limit should be independent of the path taken to $(0,0)$. Let's choose Path 1 to be the path $y = 0$, $x \to 0^+$. The limit along this path is:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0^+} \frac{x(0)}{x^2+0^2} = \lim_{x\to 0^+} \frac{0}{x^2} = 0$$

Let's choose Path 2 to be the path $y = x$, $x \to 0^+$. The limit along this path is:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0^+} \frac{x(x)}{x^2+(x)^2} = \lim_{x\to 0^+} \frac{x^2}{2x^2} = \frac{1}{2}$$

Thus, since we get two different limits along two different paths to $(0,0)$, the limit **does not exist**.

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Problem 6 Solution

6. For the function $f(x, y) = e^{2x} \cos(y)$, find the partial derivatives f_y, f_{xy}, and f_{yy}.

Solution: The first partial derivatives of $f(x, y)$ are

$$f_x = 2e^{2x} \cos(y)$$
$$f_y = -e^{2x} \sin(y)$$

The second partial derivative f_{xy} is

$$f_{xy} = (f_x)_y$$
$$f_{xy} = \frac{\partial}{\partial y} \left(2e^{2x} \cos(y)\right)$$
$$f_{xy} = -2e^{2x} \sin(y)$$

The second partial derivative f_{yy} is

$$f_{yy} = (f_y)_y$$
$$f_{yy} = \frac{\partial}{\partial y} \left(-e^{2x} \sin(y)\right)$$
$$f_{yy} = -e^{2x} \cos(y)$$

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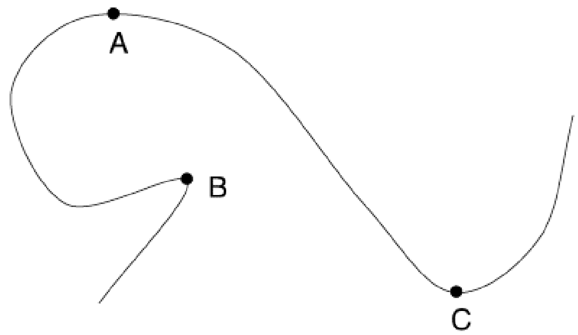
Problem 7 Solution

7. Points A, B and C are marked on the curve shown below. At which of these points is the curvature greatest?



Solution: Curvature is defined as the rate of change of the unit tangent vector $\vec{\mathbf{T}}$ with respect to arclength s.

$$\kappa = \left\| \frac{d\vec{\mathbf{T}}}{ds} \right\|$$

By inspection it appears that $\vec{\mathbf{T}}$, which is parallel to the line tangent to the curve, is changing most rapidly at point B.