Math 210, Exam 2, Fall 2011
Problem 1 Solution

1. Let $f(x, y) = y\cos(x^2y)$. Compute the directional derivative at $P = (0, 0)$ in the direction $\hat{\mathbf{u}} = \langle 0, 1\rangle$. Find the direction of the steepest descent at P.

Solution: By definition, the directional derivative of $f(x, y)$ at the point (a, b) in the direction of the unit vector $\hat{\mathbf{u}}$ is given by the formula

$$D_{\hat{\mathbf{u}}}f(a, b) = \vec{\nabla} f(a, b) \bullet \hat{\mathbf{u}}$$

The gradient of $f(x, y)$ is

$$\vec{\nabla} f(x, y) = \langle f_x, f_y\rangle = \left\langle -2xy^2\sin(x^2y), \cos(x^2y) - x^2y\sin(x^2y)\right\rangle$$

At the point $P = (0, 0)$ we have

$$\vec{\nabla} f(0, 0) = \langle 0, 1\rangle$$

Therefore, the directional derivative is

$$\boxed{D_{\hat{\mathbf{u}}}f(0, 0) = \langle 0, 1\rangle \bullet \langle 0, 1\rangle = 1}$$

The direction of steepest descent for $f(x, y)$ at (a, b) is

$$\hat{\mathbf{v}} = -\frac{\vec{\nabla} f(a, b)}{\left|\left|\vec{\nabla} f(a, b)\right|\right|}$$

Thus, at the point $(0, 0)$ the direction of steepest descent is

$$\boxed{\hat{\mathbf{v}} = -\frac{\langle 0, 1\rangle}{||\langle 0, 1\rangle||} = \langle 0, 1\rangle}$$

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Problem 4 Solution

4. Find and classify all local extrema of the function $f(x,y) = xye^{y-x}$ on the plane.

Solution: By definition, an interior point (a,b) in the domain of f is a **critical point** of f if either

(1) $f_x(a,b) = f_y(a,b) = 0$, or

(2) one (or both) of f_x or f_y does not exist at (a,b).

The partial derivatives of $f(x,y) = xye^{y-x}$ are $f_x = ye^{y-x} - xye^{y-x}$ and $f_y = xe^{y-x} + xye^{y-x}$. These derivatives exist for all (x,y) in $\mathbb{R}^2$. Thus, the critical points of f are the solutions to the system of equations:

$$f_x = ye^{y-x} - xye^{y-x} = 0 \qquad (1)$$
$$f_y = xe^{y-x} + xye^{y-x} = 0 \qquad (2)$$

Factoring Equation (1) gives us:

$$\begin{aligned} ye^{y-x}(1-x) &= 0 \\ y(1-x) &= 0 \\ y = 0, \text{ or } x &= 1 \end{aligned}$$

Note that because e^{y-x} can never be zero we can divide both sides of the equation by it. We can do the same with Equation (2) to simplify it further:

$$\begin{aligned} xe^{y-x} + xye^{y-x} &= 0 \\ xe^{y-x}(1+y) &= 0 \\ x(1+y) &= 0. \qquad (3) \end{aligned}$$

If $y = 0$ then Equation (3) gives us $x = 0$. If $x = 1$ then Equation (3) gives us $y = -1$. Thus, the critical points are $\boxed{(0,0)}$ and $\boxed{(1,-1)}$.

We now use the **Second Derivative Test** to classify the critical points. The second derivatives of f are:

$$f_{xx} = ye^{y-x}(x-2), \quad f_{yy} = xe^{y-x}(y+2), \quad f_{xy} = -e^{y-x}(x-1)(y+1)$$

The discriminant function $D(x,y)$ is then:

$$\begin{aligned} D(x,y) &= f_{xx}f_{yy} - f_{xy}^2 \\ D(x,y) &= -e^{2(y-x)}\left[x^2 - 2x + (y+1)^2\right] \end{aligned}$$

The values of $D(x,y)$ at the critical points and the conclusions of the Second Derivative Test are shown in the table below.

(a,b)	$D(a,b)$	$f_{xx}(a,b)$	**Conclusion**
$(0,0)$	-1	0	Saddle Point
$(1,-1)$	$\frac{1}{e^4}$	$\frac{1}{e^2}$	Local Minimum

Recall that (a,b) is a saddle point if $D(a,b) < 0$ and that (a,b) corresponds to a local minimum of f if $D(a,b) > 0$ and $f_{xx}(a,b) > 0$.

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Problem 3 Solution

3. Find the maximum and minimum values of the function $f(x,y) = 2x^2 + 3y^2 + 1$, where x and y lie on the ellipse $4x^2 + y^2 - 4 = 0$.

Solution: We find the minimum and maximum using the method of **Lagrange Multipliers**. First, we recognize that $4x^2 + y^2 - 4 = 0$ is compact and that f is continuous at all points on the ellipse, guaranteeing the existence of absolute extrema of f. Then, let $g(x,y) = 4x^2 + y^2 - 4$. We look for solutions to the following system of equations:

$$f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad g(x,y) = 0$$

which, when applied to our functions f and g, give us:

$$\begin{aligned} 4x &= \lambda\,(8x) \qquad (1)\\ 6y &= \lambda\,(2y) \qquad (2)\\ 4x^2 + y^2 &= 4 \qquad (3) \end{aligned}$$

We begin by noting that Equation (1) gives us:

$$\begin{aligned} 4x &= \lambda(8x)\\ 4x - \lambda(8x) &= 0\\ 4x(1 - 2\lambda) &= 0 \end{aligned}$$

From this equation we either have $x = 0$ or $\lambda = \frac{1}{2}$. Let's consider each case separately.

Case 1: Let $x = 0$. We find the corresponding y-values using Equation (3).

$$\begin{aligned} 4x^2 + y^2 &= 4\\ 0^2 + y^2 &= 4\\ y^2 &= 4\\ y &= \pm 2 \end{aligned}$$

Thus, the points of interest are $(0,2)$ and $(0,-2)$.

Case 2: Let $\lambda = \frac{1}{2}$. Plugging this into Equation (2) we get:

$$\begin{aligned} 6y &= \lambda(2y)\\ 6y &= \frac{1}{2}(2y)\\ 5y &= 0\\ y &= 0 \end{aligned}$$

We find the corresponding x-values using Equation (3).

$$\begin{aligned} 4x^2 + y^2 &= 4 \\ 4x^2 + 0^2 &= 4 \\ 4x^2 &= 4 \\ x &= \pm 1 \end{aligned}$$

Thus, the points of interest are $(1, 0)$ and $(-1, 0)$.

We now evaluate $f(x, y) = 2x^2 + 3y^2 + 1$ at each point of interest obtained in Cases 1 and 2.

$$\begin{aligned} f(0, 2) &= 13 \\ f(0, -2) &= 13 \\ f(1, 0) &= 3 \\ f(-1, 0) &= 3 \end{aligned}$$

From the values above we observe that f attains an absolute maximum of 13 and an absolute minimum of 3.

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Problem 4 Solution

4. Evaluate the following integral by reversing the order of integration:

$$\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \sin\left(x^2\right)\, dx\, dy.$$

Solution: The domain of integration is

$$D = \{(x,y) \,:\, y \le x \le \sqrt{\pi},\ 0 \le y \le \sqrt{\pi}\}$$

which can be rewritten as

$$D = \{(x,y) \,:\, 0 \le y \le x,\ 0 \le x \le \sqrt{\pi}\}.$$

Switching the order of integration and evaluating we find that

$$\begin{aligned}
\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \sin\left(x^2\right)\, dx\, dy &= \int_0^{\sqrt{\pi}} \int_0^{x} \sin\left(x^2\right)\, dy\, dx, \\
&= \int_0^{\sqrt{\pi}} \left[y \sin\left(x^2\right)\right]_0^x\, dx, \\
&= \int_0^{\sqrt{\pi}} x \sin\left(x^2\right)\, dx, \\
&= \left[-\frac{1}{2}\cos\left(x^2\right)\right]_0^{\sqrt{\pi}}, \\
&= -\frac{1}{2}\cos(\pi) + \frac{1}{2}\cos(0), \\
&= \boxed{1}.
\end{aligned}$$

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Problem 5 Solution

5. Find the mass of the solid cylinder $\mathcal{D} = \{(x, y, z) \,|\, x^2 + y^2 \leq 9,\ 0 \leq z \leq 1\}$ with density function $f(x, y, z) = 3 - 2z$.

Solution: The mass is computed via the formula

$$\text{mass} = \iiint_{\mathcal{D}} f(x, y, z)\, dV$$

We compute the integral using cylindrical coordinates. The region $\mathcal{D}$ can be described as

$$\mathcal{D} = \{(r, \theta, z) \,|\, 0 \leq r \leq 3,\ 0 \leq z \leq 1,\ 0 \leq \theta \leq 2\pi\}$$

Thus, the mass of the cylinder is

$$\begin{aligned}
\text{mass} &= \iiint_{\mathcal{D}} f(x, y, z)\, dV \\
&= \int_0^{2\pi} \int_0^3 \int_0^1 (3 - 2z)\, r\, dz\, dr\, d\theta, \\
&= 2\pi \int_0^3 r \left[3z - z^2\right]_0^1 dr, \\
&= 2\pi \int_0^3 2r\, dr, \\
&= 2\pi \left[r^2\right]_0^3, \\
&= \boxed{18\pi}
\end{aligned}$$