Grafting of Riemann Surfaces

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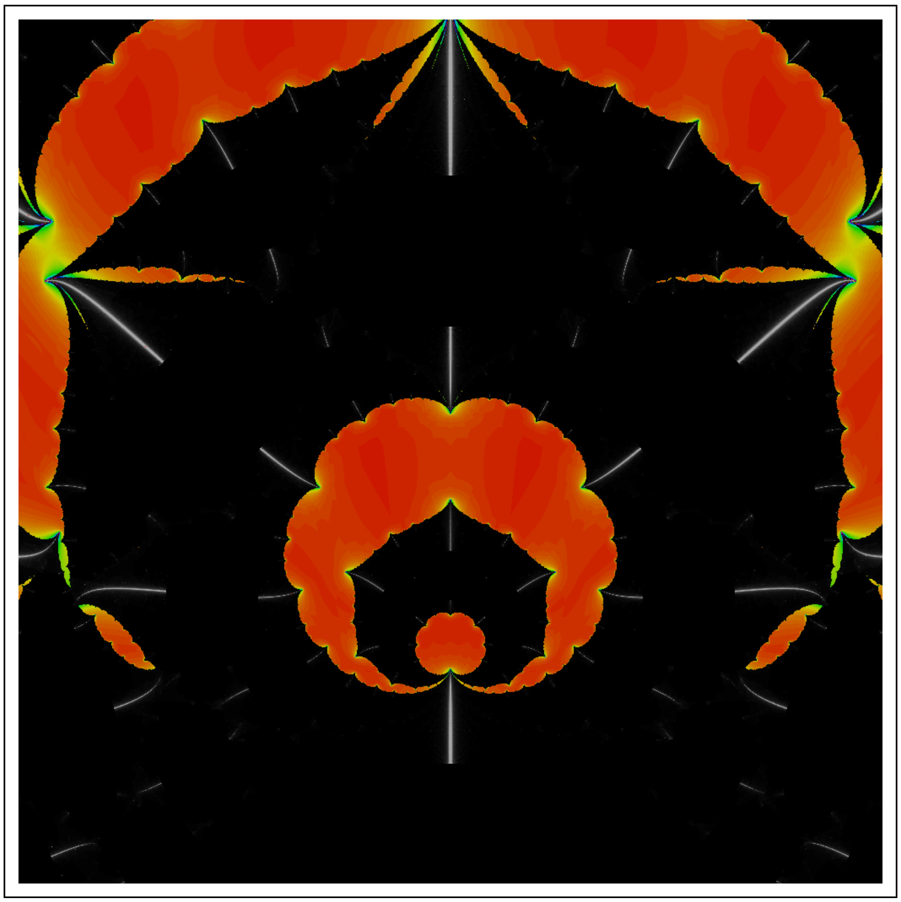
Limits of Complex Projective Structures



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Motivation

Study complex projective geometry on Riemann surfaces and connections to Teichmüller theory and three-dimensional hyperbolic geometry.

Specifically, investigate the dual nature of complex projective surfaces as both holomorphic and geometric ($\mathbb{H}^3$) objects, and try to compare these perspectives.

Results include:

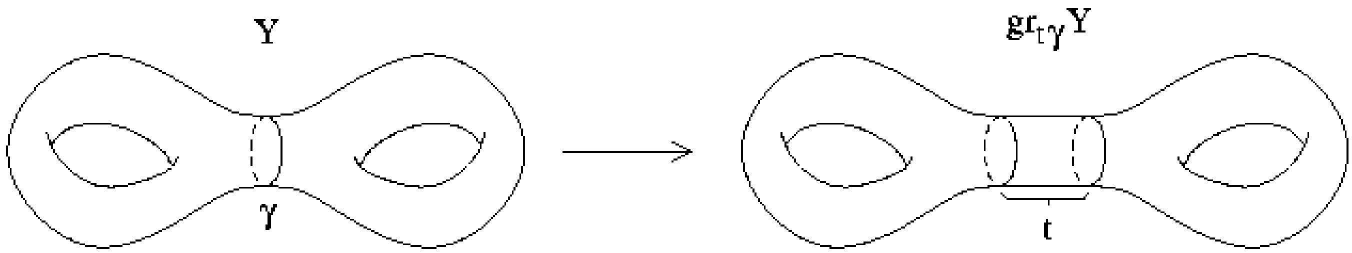
- Asymptotic formula for the change from complex-analytic to geometric coordinates.
- Geometric compactification of the deformation space of projective structures that is compatible with the foliation by underlying conformal structure.

Plan

- Grafting
- Thurston's theorem
- Complex Structures
- Limits of $\mathbb{C}P^1$ structures

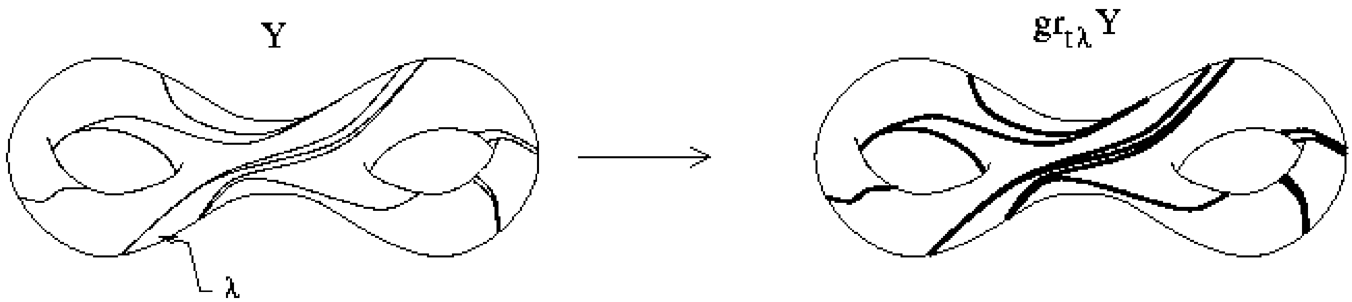
Grafting is a cut-and-paste operation on Riemann surfaces.

Start with Y, a hyperbolic RS, and γ, a simple closed hyperbolic geodesic. Now cut Y along γ and insert a Euclidean cylinder of length t.

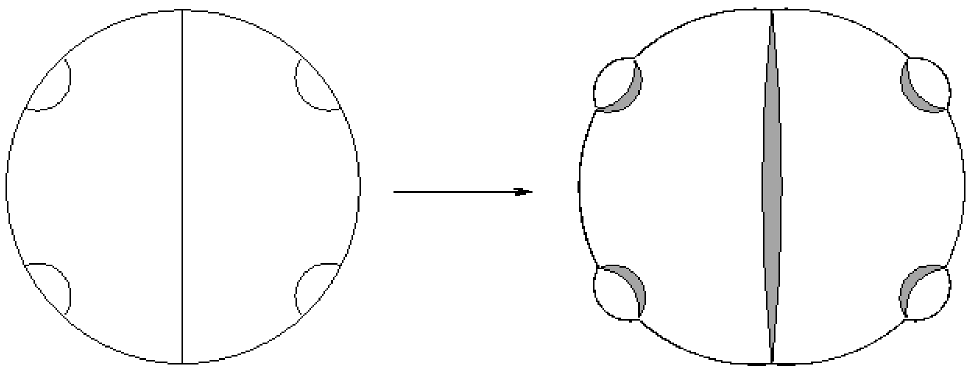


The result is $\text{gr}_{t\gamma} Y$, the grafting of Y along $t\gamma$.

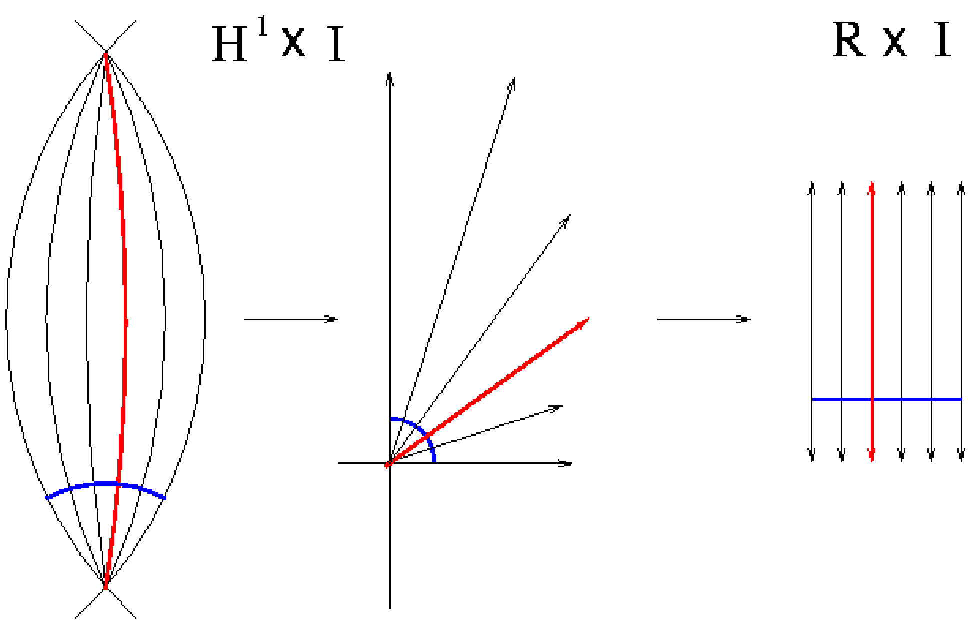
Grafting extends continuously to limits of weighted simple closed geodesics, i.e. **measured laminations**. Intuitively, cut Y along $\lambda \in \mathscr{ML}$ and insert Euclidean strips along leaves of λ.



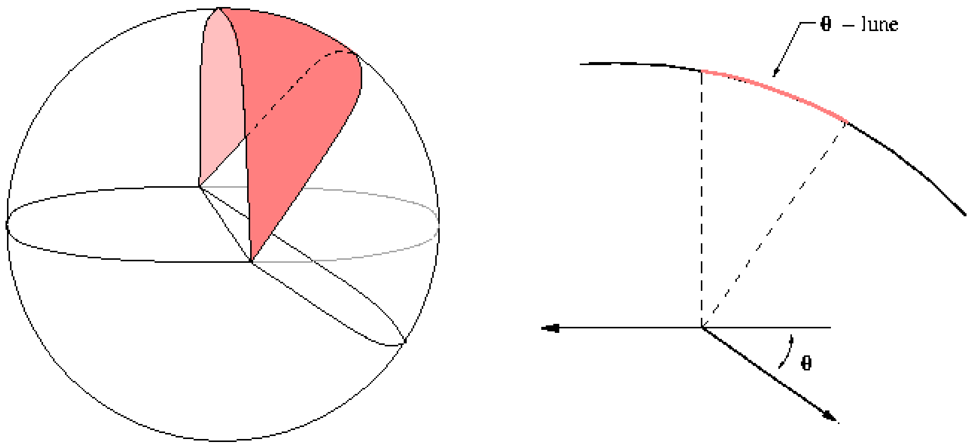
In the universal cover, this amounts to removing the lifts of geodesics in λ and replacing them with **lunes** (regions bounded by circular arcs),



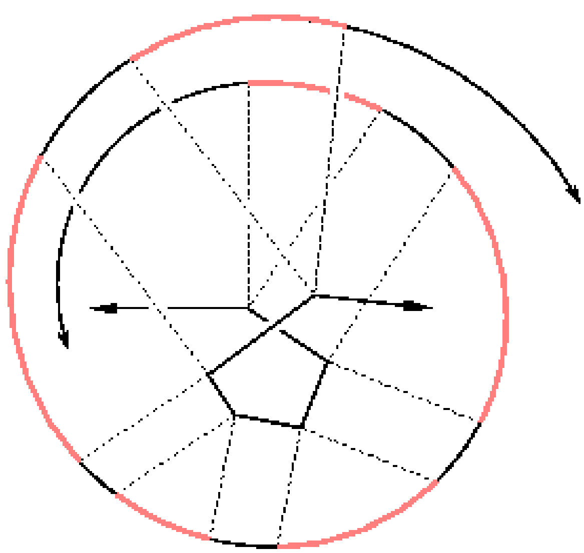
because the Euclidean metric on the lunes is the product of hyperbolic arc length and angle measure:



The relationship between $\text{gr}_\lambda Y$ and Y generalizes that between a domain $\Omega \subset \widehat{\mathbb{C}}$ and its convex hull boundary $\partial\text{CH}(\Omega) \subset \mathbb{H}^3$.



The universal cover of $\text{gr}_\lambda Y$ naturally spreads out over $\widehat{\mathbb{C}}$, which is the boundary of hyperbolic space $\mathbb{H}^3$. The boundary of the **local convex hull** is a locally convex **pleated surface** isometric to $\tilde{Y}$ bent along λ.



This map $f : \widetilde{\mathrm{gr}_\lambda Y} \to \widehat{\mathbb{C}}$ is uniquely determined up to composition with Möbius transformations, and intertwines the action of $\pi_1 Y \simeq \pi_1(\mathrm{gr}_\lambda Y)$ by deck transformations with the action of some representation $\eta(Y, \lambda) : \pi_1(Y) \to \mathrm{PSL}_2(\mathbb{C})$.

The representation $\eta(Y, \lambda)$ is sometimes called the **bending** of the Fuchsian group $\pi_1(Y)$.

The pair (f, η) define a **complex projective structure**, that is, an atlas of charts with values in $\widehat{\mathbb{C}}$ and Möbius transition functions.

Note: this is a projective structure on $\mathrm{gr}_\lambda Y$, *not* on Y. (This issue causes some headaches.)

Thurston has shown that every complex projective structure arises from grafting in a unique way, i.e. that

$$\mathrm{Gr} : \mathscr{ML}(S) \times \mathscr{T}(S) \to \mathscr{P}(S)$$

is a homeomorphism, where $\mathscr{T}(S)$ is the Teichmüller space, and $\mathscr{P}(S)$ is the space of (marked) complex projective structures.

The convex hull construction is the cornerstone of the proof; starting with a projective structure, one can then find the bending lamination and construct the inverse map Gr^{-1}.

Using this **Thurston parameterization** of $\mathscr{P}(S) \simeq \mathscr{ML}(S) \times \mathscr{T}(S)$, one can define a map

$$u : \mathscr{P}(S) \to \mathscr{T}(S)$$

that "forgets" the grafting coordinate. We call this the **ungrafting** or **convex hull** map.

The Thurston parameterization makes the connection between $\mathbb{C}\mathrm{P}^1$-geometry and three-dimensional hyperbolic geometry more transparent.

The fiber of the ungrafting map u over $Y \in \mathscr{T}(S)$ is the set of all projective structures obtained by grafting Y (or bending the Fuchsian group $\pi_1(Y)$).

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Since Möbius transformations are holomorphic, a complex projective structure also determines a complex structure. Thus we have the forgetful map, or **projection**

$$\pi : \mathscr{P}(S) \to \mathscr{T}(S).$$

The fiber of π over $X \in \mathscr{T}(S)$ is the set of all complex projective surfaces with underlying complex structure X.

However, it is not clear how the map π is related to the grafting coordinates for $\mathscr{P}(S)$.

Compare this to the case of Teichmüller space:

- symplectically natural coordinates (e.g. Fenchel-Nielsen)
- complex-analytic coordinates (e.g. Bers, horocyclic)
- Kähler structure – ?

For projective structures, we have:

- $\mathbb{H}^3$-natural coordinates (Thurston / grafting)
- complex-analytic coordinates (via π, . . .)
- unified perspective?

The fiber $P(X) = \pi^{-1}(X)$ of the projection to $\mathscr{T}(S)$ can be described in terms of **holomorphic quadratic differentials**.

A complex projective structure on X gives a **developing map**

$$f : \tilde{X} \simeq \mathbb{H} \to \widehat{\mathbb{C}}.$$

For example, the identity gives the complex projective structure coming from the Fuchsian uniformization of X.

The **Schwarzian derivative** $S(f)$ is a differential operator that measures the deviation of f from being Möbius; the result is a holomorphic quadratic differential $\phi \in Q(X)$:

$$\phi(z) = S(f(z)) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 dz^2$$

One can reverse this process; starting with $\phi \in Q(X)$, one can locally invert the Schwarzian (using only linear ODE!) to construct a developing map, and thus, a projective structure.

Hence $P(X)$ can be identified with the complex vector space $Q(X) \simeq \mathbb{C}^{3g-3}$.

Together, the fibers $\{P(X) \mid X \in \mathscr{T}(S)\}$ foliate $\mathscr{P}(S)$ by properly holomorphically embedded copies of $\mathbb{C}^{3g-3}$.

Question: What does the fiber $P(X)$ look like in the grafting coordinates?

Equivalently, what pairs (Y, λ) graft to give a Riemann surface isomorphic to X?

Compactify $\mathscr{P}(S) \simeq \mathscr{ML}(S) \times \mathscr{T}(S)$ by adding $P\mathscr{ML}(S) \times P\mathscr{ML}(S)$, where

- $P\mathscr{ML}(S)$ is adjoined to $\mathscr{T}(S)$ via the Thurston compactification, and
- $P\mathscr{ML}(S)$ is adjoined to $\mathscr{ML}(S)$ in the natural way.

The result is $\overline{\mathscr{P}(S)}$, the **geometric compactification**.

Bring the complex structure into play as follows: For $X \in \mathscr{T}(S)$, define an involution

$$i_X : \mathscr{ML}(S) \to \mathscr{ML}(S)$$

so that $i_X(\lambda) = \mu$ if λ and μ are measure-equivalent to the **vertical** and **horizontal** measured foliations of a holomorphic quadratic differential ϕ on X.

(Also use i_X to denote the induced involution on $P\mathscr{ML}(S)$.)

This involution depends sensitively on X, as orthogonality of foliations is dependent upon the conformal structure.

Thm: *The boundary of $P(X)$ in the geometric compactification $\overline{\mathscr{P}(S)}$ is the graph of i_X:*

$$\partial\overline{P(X)} = A_X = \{([\lambda], [i_X(\lambda)])\} = \{(\lambda, \mu) | i_X(\lambda) = \mu\}$$

One can also state this in terms of limits:

Cor: *Let $\lambda_i \in \mathscr{ML}$ be the grafting laminations of a divergent sequence in $P(X)$. If*

$$\lambda_i \to [\lambda] \in P\mathscr{ML}$$

then

$$Y_i \to [i_X(\lambda)] \in P\mathscr{ML},$$

where Y_i are the ungrafted surfaces, i.e. $\mathrm{gr}_{\lambda_i} Y_i = X$.

Note that $P\mathscr{ML}(S)$ is homeomorphic to an odd-dimensional sphere, and i_X is a fixed-point-free involution. **Thm A** also implies that the boundary of each $P(X)$ in $\overline{\mathscr{P}(S)}$ is a regularly embedded manifold

$$A_X \simeq S^{2k+1} \hookrightarrow P\mathscr{ML} \times P\mathscr{ML}.$$

However, while $P(X)$ and $P(X')$ are disjoint for distinct $X, X' \in \mathscr{T}(S)$, the boundaries A_X and $A_{X'}$ may intersect.

For example, this happens when X and X' are both "rectangular" punctured tori; in this case, the pair of foliations by parallels to the rectangle's edges always represent the same pair of laminations.

The geometry of this situation is reminiscent of a symmetric space M and its boundary ∂M. For each point $p \in M$, there is a **geodesic involution** $i_p : M \to M$. The set $M(p) \subset M \times M$ of pairs of points with midpoint p has boundary equal to the graph of i_p.

(One might say that grafting λ into Y is like taking the "midpoint" of λ and Y; fixing the result X forces λ and Y to move in opposite directions.)

One thing missing from this discussion is the Schwarzian derivative, which is a holomorphic quadratic differential naturally attached to a projective surface.

As a sequence of projective structures on X degenerates, it approaches a pair of orthogonal laminations (λ, μ). One might imagine that the Schwarzian is related to the quadratic differential whose vertical and horizontal laminations are (λ, μ).

Conjecture: *The Schwarzian derivative identifies the geometric compactification $\overline{\mathscr{P}(S)}$ with the compactification $\overline{\mathscr{Q}(S)}$ of the bundle of holomorphic quadratic differentials where $P^+Q(X)$ is adjoined to $Q(X)$.*

Sketch of the proof

- Start with a projective structure on X obtained by grafting, $X = \mathrm{gr}_\lambda Y$.

- Examine the retraction map $r : X \to Y$ that collapses the grafted part back to λ; recall X has the **Thurston metric** that combines hyperbolic and Euclidean parts.

- The **energy** of r is (one half of) the squared L^2 norm of its derivative; r is distance-nonincreasing, and always has an isometric direction, so the energy is the area.

- The hyperbolic part of X can be reassembled to get Y, hence its area is $2\pi|\chi|$.

- The Euclidean part of X is a thickened version of the lamination λ on Y, so its area is

$$\int_Y \mathrm{d}\ell \times \mathrm{d}m(\lambda) \stackrel{def}{=} \ell(\lambda, Y).$$

- $\mathscr{E}(r) = \frac{1}{2}\|\mathrm{d}r\|_2^2 = \frac{1}{2}(\ell(\lambda, Y) + 2\pi|\chi|)$

- Minsky's inequality:

$$\mathscr{E}(h : X \to Y) \geq \ell(\lambda, Y)^2/(2E(\lambda, X))$$

  where h is the harmonic (minimal energy) map.

- $E(\lambda, X)$ is the extremal length, the supremum of ℓ^2/Area over conformal metrics. Restricting to a subsurface (e.g., the Euclidean part) only increases extremal length.

$$E(\lambda, X) \leq E(\lambda, r^{-1}(\lambda)) = \ell(\lambda, Y)$$

- Substitute into Minsky's ineq:

$$\mathscr{E}(h : X \to Y) \geq \ell(\lambda, Y)^2/(2\ell(\lambda, Y)) = \frac{1}{2}\ell(\lambda, Y)$$

- Harmonic map has minimal energy, so

$$\frac{1}{2}(\ell(\lambda, Y) + 2\pi|\chi|) = \mathscr{E}(r) \geq \mathscr{E}(h) \geq \frac{1}{2}\ell(\lambda, Y)$$

- Thus r is nearly harmonic (excess energy is $O(1)$).

- As $Y \to \infty$, we can rescale the metric on $\tilde{Y}$ so that it converges (G-H) to the $\mathbb{R}$-tree T_μ, where $Y \to [\mu]$ (Paulin, Bestvina).

- After rescaling, $\mathscr{E}(r) - \mathscr{E}(h) \to 0$.

- Harmonic maps machinery (Korevaar-Schoen):

$$(\Delta\mathscr{E} \to 0) \Rightarrow (W^{1,2} \text{ convergence})$$

- The Hopf differential Φ of a harmonic map is the $(2,0)$ part of its derivative; vertical foliation is most compressed and horizontal most expanded.

- Harmonic map to the $\mathbb{R}$-tree T_μ collapses μ, so **vertical** foliation of Φ is equivalent to μ (Wolf).

- On the other hand, r compresses the direction orthogonal to λ and preserves the direction parallel to λ.

- Thus the Hopf differential of r, which converges to that of h, has **horizontal** foliation approaching λ.

- So in the limit, λ (grafting lamination) and μ (Thurston limit) are orthogonal.

$\square$