1. This problem concerns the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 3 & 0 \end{pmatrix}.$$

(a) (5 points) Compute the rank of A and the dimensions of the four subspaces. Put your answers in this table:

$\text{rank}(A)$	$\dim C(A)$	$\dim N(A)$	$\dim C(A^T)$	$\dim N(A^T)$
3	3	0	3	1

Solution: By elimination we find the echelon form of A:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 3 & 0 \end{pmatrix} \xrightarrow{\text{elim}} U = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus $\text{rank}(A) = 3$, and $\dim C(A) = \dim C(A^T) = 3$. Since A is 4×3, $\dim N(A) = 0$ and $\dim N(A^T) = 1$.

(b) (5 points) Circle the statements that are true:

i. A has full row rank

ii. $A\mathbf{x} = \mathbf{b}$ has at least one solution for any $\mathbf{b}$

iii. For some $\mathbf{b}$, $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions

iv. Changing the last row of A might change the rank of A

v. $\boxed{\text{Changing the last column of } A \text{ might change the rank of } A}$

Solution: Items (i) - (iii) are immediate from part (a). For (iv), note that the first three rows are independent, so the rank of A will be 3 no matter how the last row is changed. For (v), we could change the rank from 3 to 2 by replacing the last column with zeros.

This page is a continuation of problem 1. $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 3 & 0 \end{pmatrix}$

Parts (c) and (d) are about the linear system $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = \begin{pmatrix} -3 \\ -2 \\ 4 \\ -7 \end{pmatrix}$.

(c) (3 points) Does this system have a solution? *Circle your answer.* i. $\boxed{\text{YES}}$ ii. NO

Solution: Apply elimination to the augmented matrix $[A\ \mathbf{b}]$:

$$\begin{pmatrix} 1 & 1 & 0 & -3 \\ 1 & 2 & 1 & -2 \\ 2 & 0 & 2 & 4 \\ 1 & 3 & 0 & -7 \end{pmatrix} \xrightarrow{\text{elim}} \begin{pmatrix} 1 & 1 & 0 & -3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 4 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since the last equation became $0 = 0$ after elimination, $\mathbf{b}$ is in the column space and there is a solution.

(d) (5 points) If you answered YES to part (c), find the **general solution**. If you answered NO, find a **least squares solution**, i.e. a vector $\hat{\mathbf{x}}$ that makes $\|A\hat{\mathbf{x}} - \mathbf{b}\|$ as small as possible.

Solution: Since A has independent columns (full column rank), the solution is unique. We use back-substitution on the triangular system obtained in part c.

$$\begin{aligned} x + \ y \qquad &= -3 \\ y + \ z &= 1 \\ 4z &= 12 \end{aligned}$$

Thus $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$.

(Note that an exact solution is also a least squares solution, so you could find this same vector by solving $A^T A\mathbf{x} = A^T\mathbf{b}$ if, for example, you answered NO in part c.)

2. Decide whether each of the following matrices is positive definite or not. Clearly indicate what criteria you use in each case.

(a) (3 points) $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ Positive definite? i. $\boxed{\text{YES}}$ ii. NO

Solution: By the determinant test: $d_1 = 2$, $d_2 = 3$, $d_3 = 4$.

(b) (3 points) $\begin{pmatrix} 2 & 0 & x \\ 0 & 2 & 0 \\ x & 0 & 2 \end{pmatrix}$ Positive definite? i. YES ii. NO iii. $\boxed{\text{DEPENDS on } x}$

Solution: The determinant of the matrix is $8-2x^2$, which can be positive or negative.

(c) (3 points) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ Positive definite? i. YES ii. $\boxed{\text{NO}}$

Solution: The matrix has a repeated row and is therefore singular, hence not positive definite.

(d) (3 points) $\begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$ Positive definite? i. $\boxed{\text{YES}}$ ii. NO

Solution: By the determinant test: $d_1 = 3$, $d_2 = 6$, $d_3 = 20$.

(e) (3 points) $\begin{pmatrix} 5 & 4 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 5 \end{pmatrix}$ Positive definite? i. YES ii. $\boxed{\text{NO}}$

Solution: By the determinant test: $d_2 = -1$, so the matrix is not positive definite.

3. Consider a sequence of real numbers $y_0, y_1, y_2, \ldots$ satisfying the rule

$$y_{k+2} = y_{k+1} + 2\, y_k.$$

Such a sequence might begin $0, 1, 1, 3, 5, 11, 21, 43, 85 \ldots$.

(a) (5 points) Let $\mathbf{u}_k = \begin{pmatrix} y_{k+1} \\ y_k \end{pmatrix}$. Find a 2×2 matrix A such that $\mathbf{u}_{k+1} = A\mathbf{u}_k$.

Solution:

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

(b) (5 points) Find the eigenvalues λ_1, λ_2 of the matrix A.

Solution: The characteristic polynomial is $\det(A - \lambda I) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$ so the eigenvalues are: $\lambda_1 = 2$, $\lambda_2 = -1$.

This page is a continuation of problem 3.

(c) (5 points) Are there nonzero starting values y_0 and y_1 for which the sequence y_k decays to zero, i.e. $\lim_{k\to\infty} y_k = 0$? *Circle your answer.* i. YES ii. $\boxed{\text{NO}}$

If you answered YES, give an example of such starting values for y_0 and y_1. If you answered NO, explain why such decay is impossible.

Solution: The matrix A is diagonalizable and all of its eigenvalues satisfy $|\lambda| \geq 1$, so there are no solutions that decay. (The eigenvalue $\lambda_2 = -1$ corresponds to an oscillatory solution that neither grows nor decays.)

(d) (5 points) Suppose the sequence begins with $y_0 = 1$ and $y_1 = 2$. Find an exact formula for y_k.

Solution: Since $\mathbf{u}_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector of A with eigenvalue 2, we have

$$\mathbf{u}_k = 2^k \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

and thus

$$y_k = 2^k.$$

4. This question concerns the matrix

$$B = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 2c \\ 0 & 0 & c \end{pmatrix}$$

(a) (5 points) For what values of c are the columns of B linearly **dependent**?

Solution: The columns of B are linearly dependent when B is singular. Since $\det(B) = -2c$, this happens exactly when $c = 0$.

(b) (5 points) Apply the Gram-Schmidt algorithm to the columns of B to find an orthonormal basis for their span when $c = 3$.

Solution: First note that the columns of B are independent when $c = 3$, so we apply Gram-Schmidt to all three columns.

Since the first two columns are already orthogonal, the vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ are unit vectors in the same directions:

$$\mathbf{q}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathbf{q}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Since $\mathbf{q}_1$ and $\mathbf{q}_2$ span the xy plane, the third orthogonal vector $\mathbf{A}_3$ is the projection of $\begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix}$ onto the z axis:

$$\mathbf{A}_3 = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

We normalize its length to find $\mathbf{q}_3$:

$$\mathbf{q}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

This page is a continuation of problem 4.

(c) (5 points) Find the matrix R such that $B = QR$, where Q is the matrix whose columns are the orthonormal basis vectors $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ from part (b) and $c = 3$.

Solution:

$$R = Q^T B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 6 \\ 0 & 0 & 3 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & 0 & 8 \\ 0 & 2 & -4 \\ 0 & 0 & 3\sqrt{2} \end{pmatrix}$$

5. Let P_3 denote the vector space of polynomials in one variable of degree at most 3 (with real coefficients). Consider the two linear transformations

$$Id : P_3 \to P_3 \text{ defined by } Id(f) = f$$
$$T : P_3 \to P_3 \text{ defined by } T(f) = \frac{df}{dx}$$

Thus for example $Id(1+3x^3) = 1+3x^3$ and $T(1+3x^3) = 9x^2$.

(a) (5 points) Find the matrix of T using $\mathbf{v}_0 \dots \mathbf{v}_3$ as both the input basis and the output basis, where

$$\mathbf{v}_0 = 1 \quad \mathbf{v}_1 = x \quad \mathbf{v}_2 = x^2 \quad \mathbf{v}_3 = x^3$$

Solution:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) (5 points) Find the matrix of T using $\mathbf{w}_0 \dots \mathbf{w}_3$ as both the input basis and the output basis, where

$$\mathbf{w}_0 = 1 \quad \mathbf{w}_1 = 1+x \quad \mathbf{w}_2 = (1+x)^2 = 1+2x+x^2 \quad \mathbf{w}_3 = (1+x)^3 = 1+3x+3x^2+x^3.$$

Solution:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This page is a continuation of problem 5.

(c) (5 points) Find the matrix of Id using $\mathbf{w}_0 \dots \mathbf{w}_3$ as the input basis and $\mathbf{v}_0 \dots \mathbf{v}_3$ as the output basis.

Solution:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

6. Suppose S is a 3×3 **symmetric** matrix of **rank 2**, and

$$S \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

(a) (5 points) Give an example of such a matrix that is **not** a projection.

Solution: Recall that a symmetric matrix S is a projection if $S^2 = S$.
There are many possible answers, including

$$S = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which are not projections because $S^2 \neq S$.

(b) (5 points) Give an example of such a matrix that **is** a projection.

Solution: There are many possible answers; the simplest is probably

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which is the projection onto the xy plane. More generally, the projection onto any plane containing the y axis would work.

(c) (5 points) Could the first column of S be $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$? *Circle your answer.* i. YES ii. $\boxed{\text{NO}}$

If you answered YES, given an example of such a matrix; if you answered NO, explain why.

Solution: Since S is symmetric, its first row would be $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ and thus

$$S \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \cdot & \cdot \\ 1 & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ \cdot \\ \cdot \end{pmatrix} \neq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

which contradicts the hypothesis on S.

7. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

Each of the following matrices has some blank entries. If it is possible to fill in the blanks and create a matrix **similar** to A, then circle YES and give an example of how you might fill in the blank entries. Otherwise circle NO and explain why it is impossible to create a matrix similar to A by filling in the blanks.

(a) (5 points) $B = \begin{pmatrix} 2 & \cdot & \cdot \\ \cdot & 4 & \cdot \\ \cdot & \cdot & 2 \end{pmatrix}$ Might be similar to A? i. YES ii. $\boxed{\text{NO}}$

Solution: The trace of B is 8, while that of A is 9, so they cannot be similar.

(b) (5 points) $C = \begin{pmatrix} 3 & \cdot & \cdot \\ \cdot & 3 & \cdot \\ \cdot & \cdot & 3 \end{pmatrix}$ Might be similar to A? i. $\boxed{\text{YES}}$ ii. NO

Solution: Let

$$C = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Then C has eigenvalues 1, 3, and 5, and thus it is similar to A (by diagonalization).

8. Let F denote the subspace of $\mathbb{R}^4$ consisting of vectors $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ satisfying $x - y = z - w$.

(a) (4 points) Find a basis for $F^\perp$.

Solution: Note that F is the null space of $A = \begin{pmatrix} 1 & -1 & -1 & 1 \end{pmatrix}$, so $F^\perp$ is the row space of A. A basis is therefore

$$\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

(b) (4 points) Find a basis for F.

Solution: Since $F = N(A)$ and A is already in reduced row echelon form, we can read off the null space matrix:

$$N = \begin{pmatrix} -F \\ I \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A basis for F is therefore

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

(There are of course many other bases. Any three linearly independent vectors in F will work.)

This page is a continuation of problem 8.

(c) (4 points) Find the projection of $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$ onto $F^\perp$.

Solution: Since $\mathbf{b}$ belongs to F, its projection to $F^\perp$ is zero.

$$\mathbf{p} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(d) (4 points) Find the projection of $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$ onto F.

Solution: Since $\mathbf{b}$ belongs to F, its projection to F is $\mathbf{b}$:

$$\mathbf{p} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$$